Electromagnetic Induction

8.1 Magnetic Flux Φ_B

Electromagnetic induction is a phenomenon that links magnetism and electricity. While we are oversimplifying slightly here, we can say in short: Whenever or wherever a magnetic field ***B*** is CHANGING, an electric field ***E*** is induced (or "is created" or "is brought into being") in the surrounding space. If a conductor happens to be present in that region, the ***E*** field manifests as (or "becomes observable as") an *induced* emf $\mathcal{E}$, which drives an *induced* current *I* in the conductor. If there ISN'T a conductor in the area, an ***E*** field is still induced; it just isn't as easy to document. For now, let's confine ourselves to cases where there IS a conductor in the area like, say, a continuous, circular loop of wire.

To go further, we need to discuss **magnetic flux** Φ_B, the unit for which is the Weber (Wb). Recall that the concept of electric flux Φ_E quantifies the number of ***E*** field lines that pierce a (usually imaginary) surface. The ***E*** field lines (or "lines of electric flux") emanate outward from (+) charges and zero in on (–) charges. Back in Unit 2, to properly quantify electric flux Φ_E, we placed our closed, Gaussian surface so that it was perpendicular to the ***E*** field lines and was surrounding some amount of net charge *q*. By Gauss's law, if the net charge *q* inside the closed surface was (+), we said there was a net electric flux Φ_E OUT of the surface (and that it was a $+\Phi_E$); if the net charge *q* inside the closed surface was (–), we said there was a net electric flux Φ_E INTO the surface (and that it was a $-\Phi_E$). If there was zero net charge within the Gaussian surface, then there was zero electric flux Φ_E through it and, by Gauss's law, we showed that the AVERAGE ***E*** field over the entire surface was…zero, although it might NOT have been zero at any particular location.

We will do something similar here, with magnetic flux Φ_B, which is a quantitative measure of the number of ***B*** field lines that pierce a surface. These surfaces, like Gaussian surfaces, will generally be imaginary but, unlike Gaussian surfaces, which were 3-D and closed (e.g., spheres), these will be 2-D and open (e.g., circles or squares), having no "inside" and no "outside." We will choose such a 2-D surface and place it so that some quantity of magnetic field lines (i.e., "lines of magnetic flux") will pierce the surface. Let's see how good your imagination is...

Imagine our solar system: the Sun in the middle, with the planets orbiting around at various distances. Mentally place all of the planets on the same horizontal plane (which is basically true for our actual solar system, by the way). In this analogy, the orbital paths of the planets represent the continually-circling ***B*** field lines. Now imagine, from down below the plane, a vertical wire going upward, poking directly through the Sun, and going up and away from the planet-plane. Imagine an electric current *I* going in the upward direction through that wire. By the Thumbkin-Twistin' right-hand rule, with your thumb pointing in the direction of the current *I*, the ***B*** field lines would be going in concentric circles around the Sun, tracing paths very much like the orbital paths of the planets.

Now imagine a giant loop in outer space, like a big hula hoop or basketball rim. If you make it large enough and tip it sideways, at right angles to Earth's orbital path, you could make the Earth go right through it, like swishing a basketball through a hoop, but with the rim of the basket tipped sideways and the ball traveling horizontally. If you made the space-loop bigger, both Earth and Venus could swish through the loop; bigger yet, and you could make any number of planets swish through the space-loop.

This is what we are getting at with regard to magnetic flux Φ_B: How many ***B*** field lines can we "swish" through a given loop, which is the boundary of an imaginary, 2-D "surface"? Have you ever used a large (usually plastic) loop to make a soap bubble? Before you wave your arm to make the actual bubble, when the soapy film is still flat and covering the loop's opening…THAT FILM is the imaginary "surface" that we are talking about, being pierced by some number of "planetary" ***B*** field lines.

Keeping in mind the outer-space-loop analogy, you should realize that the following three factors will all affect the amount of magnetic flux Φ_B piercing the imaginary bubble-film-surface of the space-loop:

1. the size of the loop: If the center of the loop stays in the same spot, a bigger loop will allow more planets to pass through, which is equivalent to more magnetic flux Φ_B.
2. the angle of the loop: The most magnetic flux Φ_B will pass through the loop when it is oriented perpendicular to the orbital paths of the planets. If we tip the loop such that the entire loop is in the planet-plane, zero magnetic flux Φ_B can pass through the loop. (This is analogous to never being able to make a basket if the plane of the rim and the trajectory of the ball are exactly parallel to each other.)
3. how close the loop is to the Sun: Even if the size and orientation of the loop remain constant, a loop closer to the Sun experiences more flux Φ_B than does the same loop farther away. This is because the magnetic field ***B*** is stronger closer to the Sun (i.e., closer to the current-carrying wire) than it is farther away from the Sun: The ***B*** field lines are more densely spaced near the Sun-wire, just as the planets nearer the Sun are actually more-closely spaced than are the planets farther out in the solar system.

Mathematically, the magnetic flux Φ_B through a surface is defined by the surface integral

$$\varphi_B = \int \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}}$$

where $d\boldsymbol{A}$ is a "tiny-bit-of-an-area vector"; it is the same quantity we met back in Unit 2 when we learned about Gauss's law. This tiny-bit-of-an-area vector $d\boldsymbol{A}$ has the magnitude dA and a direction that is perpendicular (i.e., normal) to its surface, like an arrow embedded at right angles into a wooden battle shield. By convention, the direction of $d\boldsymbol{A}$ is taken to be OUTWARD from the $d\boldsymbol{A}$-shield, and the sign of the magnetic flux Φ_B depends on whether the ***B*** and $d\boldsymbol{A}$ vectors are parallel ($+\Phi_B$) or antiparallel ($-\Phi_B$). But the main thing that interests us is NOT the sign, but rather the MAGNITUDE of the magnetic flux Φ_B.

If the ***B*** field is uniform over the entire area *A*, then we can pull it outside the integral and, taking into account the angle θ between the ***B*** field lines and the normal-arrow-embedded-in-the-shield, we get:

$$\varphi_B = BA\cos\theta$$

Obviously, if ***B*** is NOT constant over the area *A*, we must evaluate the integral on the previous page.

The two equations given so far relate to the magnetic flux Φ_B through a single loop. If we have a coil with a total of *N* loops (or "turns"), then the total flux Φ_B for the constant-***B***-field equation becomes:

$$\varphi_B = NBA\cos\theta$$

The above two equations also demonstrate that the unit for Φ_B, the weber (Wb), is the same as a $T{\cdot}m^2$.

8.2 Faraday's Law of Induction

Each of the three points mentioned near the end of Section 8.1 suggest to us some ways in which we might CHANGE the magnetic flux Φ_B that passes through a loop. We could:

1. change the size of the loop
2. change the orientation/angle of the loop relative to the ***B*** field lines,

   e.g., spin the loop on an axis so that there is a cyclical variation in the amount of Φ_B that passes through the loop (which is exactly what is done in electrical generators)

3. move the loop to a new location where the ***B*** field is different than it was before.

In each case, the number of ***B*** field lines (i.e., the amount of magnetic flux Φ_B) that pass through the loop will be different than it was before.

And there are two more ways we can change the magnetic flux Φ_B that passes through a loop:

4. by changing the direction of the ***B*** field lines through the loop, e.g., by making the ***B*** field lines go exactly opposite to the way they used to be going. In our hoop-and-planet analogy, this would be like the planets suddenly reversing the direction of their orbital paths.
5. by straight-up increasing or decreasing the strength of the ***B*** field over time. The hoop-and-planet analogy breaks down here, but hopefully you agree that this change – YES – would indeed cause the magnetic flux Φ_B that is passing through the loop to change.

And, if a magnetic flux Φ_B is changing with time in some region of space, an electric field ***E*** "magically" (?) springs into being. ("Whoa….Groovy.") This fact – that a CHANGING magnetic flux Φ_B (let's write it as $d\Phi_B/dt$) gives rise to an *induced* electric field ***E*** – is called **Faraday's law of induction** (or sometimes, just **Faraday's law**). Let's get into Faradays' law…

Suppose we have a loop of wire, and suppose that the imaginary soap-bubble-film of the loop is (somehow) experiencing a change in magnetic flux ($d\Phi_B/dt$). Faraday's law states that an electric field ***E*** will be induced along (i.e., parallel to) each point on our loop. As you recall from Unit 6, an ***E*** field running parallel to a conducting wire gives rise to an electromotive force $\mathcal{E}$, which (of course!) generates a current *I* in the wire. So here it is, Faraday's law of induction:

$$\varepsilon = \oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -\frac{d\varphi_B}{dt}$$

This is the equation for the emf $\mathcal{E}$ induced in a single loop. If we have a coil with *N* turns, then the induced emf $\mathcal{E}$ will be proportionately larger by a factor of *N*. The REASON for this is that EACH TURN, you see, has an imaginary soap-bubble-film, and hopefully it makes sense to you that if the flux changes through EACH of the *N* films, then the induced emf $\mathcal{E}$ will be *N* times larger than if we just had a single film. ☺

You can see from the equation that Faraday's law states that both the emf $\mathcal{E}$ induced in a loop of wire – AND the induced electric field $\boldsymbol{E}$ that CAUSED that induced emf $\mathcal{E}$ – are directly proportional to the time rate of change of magnetic flux $d\Phi_B/dt$ through the loop. That is, the greater the rate that the magnetic flux Φ_B is changing, the greater the induced electric field $\boldsymbol{E}$, and thus the greater the induced emf $\mathcal{E}$. You can also see, from the dot product $\boldsymbol{E} \cdot d\boldsymbol{s}$, that the induced electric field $\boldsymbol{E}$ runs PARALLEL to any conducting element $d\boldsymbol{s}$ making up the loop; that is, $\boldsymbol{E}$ runs ALONG the loop, and NOT perpendicular to it.

As an aside, note that the leftmost two portions of Faraday's law of induction are essentially identical to the equation we first met back in Section 3.3:

$$\Delta V = -\int_A^B \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$$

After all, a ΔV IS an $\mathcal{E}$, as we learned in Unit 6. We used the above equation to do all sorts of things in Units 3 and 4, such as (1) showing that the ΔV between any two points on the same equipotential is zero, (2) showing that equipotentials and $\boldsymbol{E}$ field lines are always perpendicular to each other, and (3) deriving capacitance C equations for capacitors of various geometries. (Wow!)

Anyway, one other point about the two leftmost portions of Faraday's law is that, as was already mentioned, a conducting loop is NOT necessary for an $\boldsymbol{E}$ field to be generated due to a changing flux $d\Phi_B/dt$. The path integral $\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$ can be taken over ANY closed path, even a non-material loop, and Faraday's law allows us to determine the $\boldsymbol{E}$ field that REALLY IS generated at the location of that path.

One last thing (a rather important thing, actually) about Faraday's law: Recall that ANY emf $\mathcal{E}$ (or ΔV, if you'd prefer) is an amount of work W that would be done per unit charge q. (To help convince you, this facts is even shown in the units: 1 V = 1 J / 1 C of charge.) Anyway, in Faraday's law, we have an *induced* emf $\mathcal{E}$ (i.e., a ΔV) which is equal not only to $\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$ but also equal to $-\frac{d\varphi_B}{dt}$. ("Yeah. So…?") So, the point here is that an *induced* ***E*** field differs from an electrostatic ***E*** field created by a stationary charge (i.e., every ***E*** field we've met up until today) in that an *induced* ***E*** field is NOT conservative, while an electrostatic one IS. Recall that, for things to be conservative, the work that they do over any CLOSED path must equal ZERO. In the equation of Faraday's law, any work that might be done by the induced ***E*** field is NOT equal to zero; it is equal to $-\frac{d\varphi_B}{dt}$, which is definitely NOT zero, whenever induction is happening. So, once again, while an electrostatic ***E*** field IS conservative, an induced ***E*** field is NOT.

8.3 Lenz's Law

Lenz's law is all about the (–) sign in front of the $d\Phi_B/dt$ term in Faraday's law of induction. That (–) sign deserves some explanation, so let's tackle that now.

Consider a loop of wire again, and let's say that the magnetic flux Φ_B through it is changing at some rate $d\Phi_B/dt$. By Faraday's law, an ***E*** field is set up within the wire, and it will be oriented in one of two directions: say, either clockwise (CW) or counterclockwise (CCW). The induced emf $\mathcal{E}$ will have that same CW or CCW orientation, and THAT will cause – within the wire – a current *I*…which is ALSO in that CW or CCW direction. So far, so good.

Now, you learned in the past that ANY current *I* produces a magnetic field ***B***, which means that our induced current *I* will produce an induced magnetic field ***B*** that superposes with the changing ***B*** field that caused all of this trouble in the first place. So, at long last, here's the deal with the (–) sign in Faraday's law of induction: The DIRECTION of the induced ***E*** field AND the induced emf $\mathcal{E}$ AND the induced current *I*…is such that the induced ***B*** field OPPOSES the changing magnetic flux $d\Phi_B/dt$ that was the root cause of this whole thing. In short:

1. If the magnetic flux Φ_B is decreasing, the induced ***B*** field “fights against” that decrease, trying to buoy up the flux Φ_B to be “where it was before” (but never fully succeeding).
2. If the magnetic flux Φ_B is increasing, the induced ***B*** field “fights against” that increase, trying to hold the flux Φ_B down, to “where it was before” (but never fully succeeding).

The above discussion is the essence of Lenz’s law. Here it is, formally:

Lenz’s law *states that an induced emf $\mathcal{E}$ and an induced current I in a conductor are in such a direction as to generate a magnetic flux Φ_B that opposes the change in magnetic flux $d\Phi_B/dt$ that produced them.*

The opposition stated in Lenz’s law is required by the law of conservation of energy. If the induced current generated by the changing magnetic flux $d\Phi_B/dt$ were to set up a REINFORCING flux Φ_B (rather than the actual oppositional one), then everything that follows (including the amount of energy associated with the system) would spiral all the way to an impossible limit, either all the way UP to infinity or all the way DOWN to zero. Either case would violate the law of conservation of energy.

8.4 Motional EMF

This section could perhaps benefit by the insertion of a graphic, but having a good visual-imagination definitely helps in studying electricity and magnetism, so we’re going to practice using that. Here we go...

Imagine a horizontal conducting bar of length *l* in front of you. Imagine also that there is a uniform magnetic field ***B***, also directed horizontally, but AWAY from you and TOWARD-and-through the bar. If you remember that a physics quantity directed AWAY from you is represented by a ⊗…then there are a bunch of these, evenly spaced, all around the horizontal bar. Okay, now we’re going to set things moving…

Imagine that (somehow) we start moving the bar upward at a speed *v*. Combining the equation

$\vec{\boldsymbol{F}}_{\boldsymbol{B}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$ from Unit 7 with the "Bang-bang!" version of the right-hand rule (pointer finger in direction of ***v***, tall-man in direction of ***B***, with thumbkin pointing in the direction of the magnetic force $\boldsymbol{F_B}$), you should see that there will be a magnetic force $\boldsymbol{F_B}$ directed to the LEFT on the (+) charges in the bar. Conventional (+) charge will thus move LEFT and there will immediately be a slight (+) charge on the left end of the bar and a slight (–) charge on the right end, although overall the bar will still have zero NET charge. Thus, by moving the bar upward and "cutting through" the ***B*** field lines, we have generated an *induced ΔV* across the bar, with the left end now being (+) and having a higher potential *V*, while the right end is (–), with a lower potential *V*.

However, as soon as we separate charge within the bar, with (+) on the left and (–) on the right, there is generated within the bar an ***E*** field. This ***E*** field points from high- to low potential, i.e., from (+) to (–), i.e., from left to right, and it imposes an *electric* force $\overrightarrow{\boldsymbol{F_E}} = q\vec{\boldsymbol{E}}$ TO THE RIGHT(!) on the (+) charges that were just-a-second-ago happily moving leftward due to the *magnetic* force $\boldsymbol{F_B}$. An equilibrium is established when the two forces balance:

$$\vec{\boldsymbol{F}}_{\boldsymbol{B}} = \vec{\boldsymbol{F}}_{\boldsymbol{E}} \quad so \quad q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} = q\vec{\boldsymbol{E}} \quad so \quad \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} = \vec{\boldsymbol{E}}$$

And, since everything here is perpendicular to everything else, the magnitudes are related by an equation that frequently needs to be used by examinees on the AP Physics C Exam: $E = v\,B$

The ***E*** field that we've been talking about within the bar is related to the *ΔV* across the bar (of length *l*, you recall) by an equation we've seen many times (most recently in Sec. 8.2), namely: $\Delta V = -\int_A^B \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$

Here, the ***E*** field is constant, so this magnitude turns into: $\Delta V = E\,l \quad or \quad E = \frac{\Delta V}{l}$

If we substitute this result into the $E = v\,B$ equation, we get: $\Delta V\ \ (or\ \varepsilon)\ = v\,B\,l$

The last equation above is important because it, too, often needs to be employed on the AP Physics C Exam. This equation gives the emf $\mathcal{E}$ between the ends of a conducting bar of length *l* when the bar is moved at a speed *v* in such a direction that it "cuts" the ***B*** field lines at a right angle. As crazy as it sounds, moving a conducting bar as we've been describing is actually a primitive generator: We're taking mechanical energy and changing it into an electrical form. Once more: Wires moving (in a particular way) through ***B*** fields will experience an induced emf $\mathcal{E}$ between their ends. Okay, back to our horizontal bar…

Let's change the scenario slightly: We still have a left-right horizontal bar of length *l*, but now there's also a (reasonably hefty) wire that is in a U-shape, with the bottom of the U parallel to the bar and the sides of the U pointing straight upward. Our bar of length *l* connects the vertical sides of the wire together, like this… ~~U~~. Our bar can slide freely up and down the sides of the U, in what is sometimes called a **rail system**. Make sure you understand that the bar and the bottom part of the U, taken together, form a complete conducting loop. Also, don't forget that we have a bunch of these symbols… ⊗ …everywhere around the U and the bar; these represent a uniform ***B*** field that points AWAY from us.

So now again, we begin to move the bar upward at a speed *v*. Again, a magnetic force $\boldsymbol{F_B}$ causes conventional (+) charge in the bar to flow to the left. BUT, this time, the bar is part of a continuous loop, and the result is that the charge moves at EVERY point around the loop, in the counterclockwise direction. That is, there's now a current *I* in the loop, and it's due to an induced emf $\mathcal{E}$ that is caused by our raising the bar. (Ha! Raising the bar…) Anyway, this induced emf $\mathcal{E}$ is numerically equal to $v\,B\,l$ (derived above), and IT is responsible for the counterclockwise current *I* around the loop.

Suppose now that we've raised the bar far enough and we start to push it back downward with a speed *v*. Use $\vec{\boldsymbol{F}}_{\boldsymbol{B}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$ and the "Bang-bang!" right-hand rule to convince yourself that the (+) charges in the bar will experience a magnetic force $\boldsymbol{F_B}$ to the RIGHT; we will therefore generate an induced current *I* in the CLOCKWISE direction around the loop.

Here is the last new thing about this phenomenon called **motional emf**… Think about the most recent case: the bar moving downward, with a current *I* flowing through the bar to the right. The current *I* in the bar (of length *l*, or *L*, if you wish) is still in the presence of the uniform $\boldsymbol{B}$ field (which points away from us) and is, of course, at right angles to it. By this equation from Unit 7… $\vec{\boldsymbol{F}}_{\boldsymbol{B}} = I\vec{\boldsymbol{L}} \times \vec{\boldsymbol{B}}$ …there is now a magnetic force $\boldsymbol{F_B}$ on the bar itself, and in what direction does that force act? The "pat-pat" version of the right-hand rule (fingers in direction of $\boldsymbol{B}$ field, thumbkin in direction of current *I*, with the right palm facing in the direction of the magnetic force $\boldsymbol{F_B}$) provides the answer: UPWARD! When we exert a force to move the bar downward, the current *I* that is induced generates a force $\boldsymbol{F_B}$ upward, which opposes the downward motion. This effect is a variation on Lenz's law: Whenever we try to change something, Nature attempts to cancel it out and maintain an equilibrium.

If you care to, mentally go through the results you'd get if the $\boldsymbol{B}$ field were instead pointing TOWARD you, like this… ⊙ . Figure out the direction of current *I* around the loop when you raise the bar; then use the $\vec{\boldsymbol{F}}_{\boldsymbol{B}} = I\vec{\boldsymbol{L}} \times \vec{\boldsymbol{B}}$ equation to figure out the direction of the induced $\boldsymbol{F_B}$ force on the bar. Repeat the thought experiment for a lowering of the bar. Questions involving motional emf are very common on the national Exam for AP Physics C, so spend some time making sure you're fully on board with this Section.

8.5 Inductors and Inductance

Lenz's law is explicitly demonstrated in electrical circuits in the behavior of **inductors**. Recall that Lenz's law states that the induced magnetic field $\boldsymbol{B}$ in the space surrounding a closed loop of wire opposes, at every instant, the CHANGE in magnetic flux $d\Phi_B/dt$ (due to some external source!) that the loop "senses." In line with this, we now introduce the inductor, an electrical device that:

1. opposes all instantaneous changes in current I, BUT…
2. has NO effect on steady (i.e., unchanging) currents.

Once again, inductors resist CHANGES in currents I (which, incidentally, are due to changes in magnetic flux $d\Phi_B/dt$). Furthermore, the greater the rate-of-change in a current dI/dt, the more an inductor will resist that change. Another way of saying the same thing is that the greater the rate-of-change of current dI/dt, the greater will be the **<u>self-induced emf</u>** $\mathcal{E}$ in the inductor. This self-induced emf $\mathcal{E}$ is sometimes called **<u>back emf</u>** because it OPPOSES the change in current dI/dt.

You will be heartened to learn that an inductor is an unbelievably simple electrical device: An inductor is simply a coil of wire, with the wire usually turned many times around a cylindrical axis: a solenoid! Recall that the magnetic field $\boldsymbol{B}$ within a solenoid is uniform and runs parallel to the central axis of the solenoid. When a solenoid is connected into a circuit along with a resistor, the solenoid functions as an inductor.

Inductors have the important electrical property of **<u>inductance</u>** L (sometimes called **<u>self-inductance</u>** L) which is a measure of how much *opposition* a coil offers to a *change* in the current through it. (Inductance is symbolized by an L to honor Heinrich Emil Lenz, of Lenz's law fame.) If an inductor has a large value of inductance L, it REALLY fights against any change in current I that the inductor is experiencing. If an inductor has a small value of inductance L, it fights ONLY-A-LITTLE against any change in current I that it is experiencing. No matter how big or small the value of inductance L, an inductor will put up NO FIGHT AT ALL if the current I is constant, i.e., is NOT changing. Inductance L is essentially the electrical equivalent of inertia or mass: It really only becomes noticeable when we attempt to CHANGE things. The unit for inductance L is the henry (H), named after American physicist Joseph Henry.

Recall that the capacitance C of a capacitor depends only on its geometry, and NOT on quantities such as potential difference ΔV or amount of charge stored Q. Similarly, the inductance L of an inductor depends only on ITS geometry, and NOT on any other quantities such as the current I running through it. For a solenoid of N turns, with a length l, and having the cross-sectional area A, it can be shown that the inductance L of the solenoid is given by:

$$L = \frac{\mu_o N^2 A}{l}$$

8.6 RL Circuits

Circuits containing both resistors and inductors are called RL circuits. Like resistor-and-capacitor circuits (RC circuits), RL circuits exhibit time-dependent behavior. Changes in voltages ΔV and currents I in RL circuits are related to the **inductive time constant** τ which, for RL circuits, is equal to the inductance L divided by the resistance R:

$$\tau = \frac{L}{R}$$

As an aside, perhaps you recall that the time constant τ for RC circuits was given by: $\tau = RC$

In this class, we will deal only with direct-current (DC) RL circuits.

In our earlier studies of RC circuits, you might recall that we went through how a capacitor acts like a wire at the first instant of connection and as a perfect insulator when it is fully charged. RL circuits have inductors rather than capacitors, but inductors – like capacitors – exhibit very specific behaviors at particular points in time, namely: An inductor acts as an insulator at the first instant of connection, and it acts as a simple wire at steady state. So you can see that, in a circuit, an inductor behaves in somewhat opposite fashion to a capacitor.

Suppose we have an RL circuit where the resistor (with resistance R) and inductor (with inductance L)

are connected in series to a battery of emf $\mathcal{E}$. If the switch is closed at t = 0, the current I in the circuit increases with time according to the equation (sometimes called the equation for the "rising current"):

$$I(t) = \frac{\varepsilon}{R}\,(1 - e^{-t/\tau})$$

Remembering that $e^0 = 1$, that $e^{-\infty} = 0$, and also that inductors always put up a fight when currents I in circuits CHANGE, we can see from the above equation that:

1. The current I at the first instant of connection (t = 0) equals zero, i.e., $I(0)$ = zero. This observation corroborates our earlier claim that an inductor acts like an insulator at the first instant of connection. The circuit used to have ZERO current flowing through it and now the battery is trying to change that, and the inductor resists that change…although, in the end, it always fails.
2. The current I at a much later time, i.e., when the circuit has reached steady state, is a simple application of Ohm's law, i.e., $I(\infty) = \mathcal{E}/R$. The current I will then stay at this equilibrium value for as long as nothing else changes in the circuit. This observation corroborates our earlier claim that an inductor acts like nothing more than a simple wire when the circuit has reached the steady-state condition. Nothing is changing anymore, which is just the way an inductor likes it, and so it just curls up in the corner and goes to sleep; and it will stay like that for as long as the current I remains constant.

You might recall from our work with RC circuits that, in a time of ONE time constant τ starting at t = 0: 63% of the change-that's-going-to-happen…has happened. That is the case here, as well: The current I increases to 63% of its final (i.e., maximum) value in the first τ seconds. It climbs another 63% of the-rest-of-the-way-to-the-top in the next τ seconds.

Once the above equilibrium has been established, hopefully it is clear that the inductor is doing absolutely nothing interesting; it is behaving just like an ordinary wire. If at this point, however, the battery is suddenly removed and replaced with a wire, then the current begins to decay exponentially according to the following equation…

$$I(t) = \frac{\varepsilon}{R}\, e^{-t/\tau}$$

…where we have now re-defined time such that $t = 0$ at the first instant of the change. The “63% rule” holds here, too: In the first τ seconds, the current *I* will decrease to 37% of its maximum value. In the next τ seconds, the current *I* will decrease until it is 37%-of-the-first-37%.

Incidentally, this removing-of-the-battery-and-replacing-it-with-a-wire is most easily accomplished by constructing the circuit with a switch, whose purpose is to toggle back-and-forth between the circuit’s “rising current” configuration and its “decaying current” configuration. In any case, our sleeping inductor suddenly “wakes up” when it senses a change happening. As before, we see from the equation above that:

1. At the first instant the switch is flipped, bypassing the battery, the current *I* equals $\mathcal{E}/R$, i.e., $I(0) = \mathcal{E}/R$. That is, even though there is no longer a battery pushing charge through the circuit, the inductor acts to try to keep the current *I* flowing just as it was before. Remember, inductors fight/resist/try-to-offset/try-to-make-up-for ANY change in a circuit’s current *I*. In this case, there used to be a current *I* of $\mathcal{E}/R$ flowing through the circuit and now there isn’t, and so the inductor fights to try to keep that current going… although, in the end, it always fails.
2. At a much later time, the current *I* has dwindled to zero, i.e., $I(\infty)$ = zero. The current *I* will then stay at zero until something new happens in the circuit (like if the switch is flipped back, re-engaging the battery’s effort). While the current remains at zero, here again, nothing is changing anymore, and the inductor ceases to fight-the-good-fight and quietly goes back to sleep.

Somewhat ironic, isn't it, that an inductor will fight ANY change in current I at every incremental step along the way but, once the current has reached some new value and ceases to change anymore, the inductor is suddenly "all good" with things…?

8.7 Energy and Inductors

Hopefully you recall that potential energy U is stored in the electric field $\boldsymbol{E}$ that exists between the plates of a capacitor. One equation for computing the energy U stored in a capacitor of capacitance C, whose plates are at a potential difference ΔV is:

$$U = \frac{1}{2}CV^2 \qquad OR \qquad U = \frac{1}{2}C(\Delta V)^2$$

Energy U is also stored in inductors. This time, however, the energy U is stored NOT in the electric field $\boldsymbol{E}$, but rather in the magnetic field $\boldsymbol{B}$ (mostly within, but also around) the inductor. For an inductor of inductance L that is carrying a current I, the

potential energy U stored in its $\boldsymbol{B}$ field is given by: $U = \frac{1}{2}LI^2$

This is the energy required to establish the $\boldsymbol{B}$ field within the inductor.

You can see the similarity in the format of the potential energy U equations for capacitors and inductors. And, while we're at it, let's add more fuel to the fire by recalling from our studies in mechanics that the potential energy U for a spring with force constant k and displacement x was given by:

$$U = \frac{1}{2}kx^2$$

Yeah…pretty neat.

8.8 Self-Inductance (Revisited) vs. Mutual Inductance

What we've been talking about so far with regard to inductors is called **self-induction**. Self-induction occurs in a coil when the current I in the coil changes with time, dI/dt. We've established that an inductor resists any and all changes in the current I that flows through it. The REASON inductors resist changes in current I is because, whenever the current changes dI/dt, the magnetic flux Φ_B through the many soap-bubble-films of the inductor changes $d\Phi_B/dt$, which means – from Faraday's law – that an emf $\mathcal{E}$ is induced in those coils of the inductor. This induced emf $\mathcal{E}$, in accord with Lenz's law, is always in a direction so as to resist those original changes in the current dI/dt. We saw both of these laws in Section 8.2, but here they are again:

Faraday's law: $\varepsilon = \oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -\frac{d\varphi_B}{dt}$

Lenz's law: *An induced emf $\mathcal{E}$ and an induced current I are in such a direction as to generate a magnetic flux Φ_B that opposes the change in magnetic flux $d\Phi_B/dt$ that produced them.*

For the self-induction we've been discussing, basically here's what happens:

1. The current I in the inductor (i.e., coil, i.e., solenoid) changes, for whatever reason: somebody flipped a switch on or off, or the current happens to be an alternating current produced by an electric generator.
2. Since currents I produce magnetic fields $\boldsymbol{B}$, the changing-with-time current dI/dt produces a changing-with-time magnetic field $d\boldsymbol{B}/dt$, which of course results in a changing-with-time magnetic flux $d\Phi_B/dt$ through each of the soap-bubble-films (one for each of the N turns) of the inductor/coil/solenoid.
3. The inductor "notices" (we're personifying here) the changing magnetic flux $d\Phi_B/dt$ and, consistent with Faraday's law, an emf $\mathcal{E}$ is induced in the inductor. (That's why it's called self-induction.) The bigger/more-violent the magnitude of $d\Phi_B/dt$, the bigger/more-violent the induced emf $\mathcal{E}$.

4. Consistent with Lenz's law, the direction of the induced emf $\mathcal{E}$ is such that any effects from it will fight/oppose/resist the original changing flux $d\Phi_B/dt$, to wit: The oppositional emf $\mathcal{E}$ produces an oppositional current I that produces an oppositional magnetic field $\boldsymbol{B}$ that (of course!) opposes the original changing flux $d\Phi_B/dt$, which was caused by the original changing current dI/dt. (Whew!)

We know that the self-induced emf $\mathcal{E}$ can be found using Faraday's law $\varepsilon = \oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -\frac{d\varphi_B}{dt}$

namely, by either taking a **<u>path integral</u>** of the $\boldsymbol{E}$ field around the loop OR by knowing the rate-of-change of magnetic flux $d\Phi_B/dt$. BUT…it can also easily be found if we know the rate-of-change of the original current dI/dt, by the following relation:

$$\varepsilon = -L\frac{dI}{dt}$$

where L is the inductance of the coil. So, if you're asked to find a self-induced emf $\mathcal{E}$, use either the above equation OR Faraday's law, depending on what other information you are given in the problem.

In case you're wondering where the equation for the energy U of an inductor comes from…

Recall the power in an electric circuit: $P = I\,\Delta V$

Well, induced emf $\mathcal{E}$ is also a ΔV, so: $P = I\,\varepsilon$

Substituting in our newest expression for $\mathcal{E}$ and disregarding the (−) sign: $P = I\,L\,\frac{dI}{dt}$

A little algebraic manipulation: $P\,dt = I\,L\,dI$

If we integrate both sides, we get energy (power x time) on the left, and what else? $U = \frac{1}{2}LI^2$

Yep. Pretty neat.

While self-induction is a phenomenon demonstrated by a SINGLE coil or solenoid in a circuit, **mutual induction** describes the electromagnetic interaction between, say, A PAIR of coils. Let's think a little…

Suppose we have a conducting coil in which the current is changing. The time-varying current dI/dt produces a time-varying magnetic field $d\boldsymbol{B}/dt$ in all of the space around the coil. The amount of the changing magnetic field $d\boldsymbol{B}/dt$ – i.e., the $d\boldsymbol{B}/dt$ "bits" – that pierce the many soap-bubble-films of the coil…constitute a changing magnetic flux $d\Phi_B/dt$ for the coil. This changing magnetic flux $d\Phi_B/dt$ generates a back emf $\mathcal{E}$ in the coil, according to Faraday's law, which opposes the original dI/dt. (Blah-blah-blah…) All of this has been said previously. So far, so good.

But what about the $d\boldsymbol{B}/dt$ bits that are OUTSIDE the coil? Until now, we have neglected THESE $d\boldsymbol{B}/dt$ bits because they don't make any contribution to the back emf $\mathcal{E}$ that is induced in the coil…and that's what we've been interested in. Suppose now, however, that we bring a SECOND conducting coil near to the first one. Obviously, the bubble-films of ITS coils are in a different space than those of the first coil, but do you see how the bubble-films of the second COULD be pierced by $d\boldsymbol{B}/dt$ bits that are generated by the first? In other words, by just being near to the first coil and having some of the first coil's $d\boldsymbol{B}/dt$ bits pierce ITS OWN bubble-films, we generate – in the second coil, via Faraday's law – an emf (let's call it $\mathcal{E}_2$) in that second coil. ("Whoa, dude…") The emf $\mathcal{E}_2$ will produce a current I_2 in that second coil, and it shouldn't surprise you that $\mathcal{E}_2$ is proportional to the RATE at which the current in the first coil is changing, dI_1/dt. By Lenz's law, the induced emf $\mathcal{E}_2$ will have a direction such that whatever happens in the second coil acts in opposition to whatever is happening in the first. That's the essence of mutual induction.

We should note here that the wire used in the construction of ANY inductor should generally be insulated, i.e., it should be a metal wire with insulated sheathing. Insulated wire is necessary because the properties of inductors/coils/solenoids depend crucially the number of turns (basically, the number of bubble-films) per unit length. If the wire is NOT insulated and the adjacent turns of bare wire touch each other, then you really have ZERO turns because the current will simply skip directly from turn to turn without bothering to travel AROUND any of the loops. In a similar vein, for mutual induction, the coils must not only be made of insulated wire and be close to each other – even touching, perhaps – but they must also be insulated from each other, because each coil MUST have its own unique current I.

You recall that the variable L has been assigned to designate the self-inductance of a single inductor. When multiple inductors are placed near each other so that mutual inductance can take place, a quantity called the **<u>mutual inductance</u>** – which has been assigned the variable M – is associated with that conformation of multiple inductors. For a given conformation of inductors, a single value of mutual inductance M applies to each and every inductor in the combination. The simplest equation for calculating the mutual inductance M for a combination of two inductors of inductance L_1 and L_2 is:

$$M = \sqrt{L_1 L_2}$$

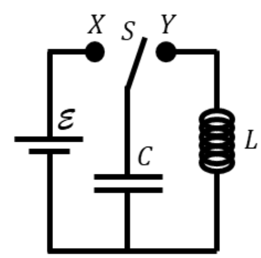
Recall that inductance L has the unit henries (H), so – above – the unit of the mutual inductance M for two inductors would be the square root of henries-squared, which is…henries (H).

The mutual inductance M of a combination of, say, two inductors allows us to relate the rate-of-change of current dI/dt in one of the coils to the induced emf $\mathcal{E}$ in the other coil; that is:

$$\varepsilon_2 = -M \frac{dI_1}{dt} \quad and \quad \varepsilon_1 = -M \frac{dI_2}{dt}$$

8.9 A Conceptual Approach to LC Circuits

As their name suggests, LC circuits are ones containing inductors and capacitors. We will deal here with the behavior of the simplest of such circuits: a circuit containing a battery, one capacitor, and one inductor. While graphics have generally been deliberately left out of this series of reading guides in order that the flow of the reading NOT be interrupted, a diagram has been included here, as this is very difficult to understand otherwise. Suffice it to say that LC circuits behave WAY differently than anything we've met up to this point, so be prepared for some funkiness…



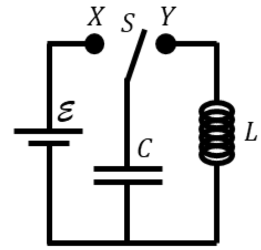
We will consider the circuit shown. Note that the switch can go back-and-forth between points *X* and *Y*. Let's walk through this unique sequence of events:

1. We first throw the switch to *X*. The capacitor charges, resulting in a charge $+Q$ on the top plate, a charge $-Q$ on the bottom plate, and a voltage ΔV between plates. Energy U is stored in the $\boldsymbol{E}$ field between plates. So far, nothing has happened with the inductor.

2. Now we throw the switch to *Y*; the battery is now effectively removed from the circuit.

3. The capacitor begins discharging, which pushes charge in a clockwise (CW) direction, away from the top plate, through point *Y*, through the inductor, and toward the bottom plate. But inductors resist any and all changes in currents (the current through the inductor used to be zero and now the discharging capacitor is trying to change that), so the current I through the inductor (and the circuit) will be ZERO for an instant, after we throw the switch to *Y*. However, inductors cannot PREVENT changes in current (they can only RESIST changes), so the current I grows steadily – with the inductor resisting each step of the way – until the current reaches some maximum value. During this current-growth period, the energy U in the $\boldsymbol{E}$ field between plates, i.e., $\frac{1}{2}C(\Delta V)^2$, is dropping (because the ΔV across the plates is

dropping), while the energy U in the inductor's $\boldsymbol{B}$ field, i.e., $\frac{1}{2}LI^2$, is increasing (because the current I is increasing). At the moment when the current I reaches a maximum in the CW direction, the capacitor has completely discharged (so Q and ΔV and $\boldsymbol{E}$ are all zero), and it has lost all the energy U that was formerly in its $\boldsymbol{E}$ field. All of that energy is now stored in the $\boldsymbol{B}$ field of the inductor.



4. At this point, the capacitor is completely discharged and has no ΔV that can push any more charge. But the inductor is "used to" a current flowing CW and, when it "senses" the CW current slackening off due to a lack of push from the capacitor, it now releases energy from its $\boldsymbol{B}$ field in an effort to keep that CW current going…which it does, with mild success, as the current does indeed continue flowing CW, although at a diminishing rate, despite the inductor's continued energy input. The CW current dwindles to a mere trickle and, finally, when the inductor has given up every single bit of the energy that was only recently stored in its $\boldsymbol{B}$ field, the current is zero. All of the energy is back within a new $\boldsymbol{E}$ field between the plates, which (because the current has been going CW this whole time) are now charged OPPOSITE to what they were originally: $+Q$ resides on the BOTTOM plate and $-Q$ on the TOP! But only momentarily…

5. What, do you suppose, happens next? Bingo: The capacitor begins to discharge again, but this time producing a current in the counterclockwise (CCW) direction. Again, the inductor resists the rising CCW current each step of the way. Again, when the CCW current reaches a maximum, the capacitor is fully discharged and all of the energy U is once again stored in the inductor's $\boldsymbol{B}$ field. Again, when the capacitor ceases to "push" current in the CCW direction, the inductor releases energy in an attempt to maintain that CCW current. Again, when the inductor's energy has been exhausted, the current has fallen to zero, the capacitor is charged as it originally was (with $+Q$ on the top plate and $-Q$ on the bottom), and all of the energy is once again stored in the capacitor's $\boldsymbol{E}$ field.

And then the sequence repeats, starting at Step 3 above, in a continuing cycle: 3-4-5, 3-4-5, 3-4-5, etc. with the energy sloshing back and forth between the capacitor's ***E*** field and the inductor's ***B*** field in what is essentially an alternating current (AC) circuit. At some point, obviously, due to thermal losses associated with all currents (except in superconductors), the sloshing dies down and eventually ceases entirely. What we have here is a circuit with a number of oscillating quantities – charge Q, current I, and so on – which vary in the same sinusoidal manner as do certain quantities for a simple harmonic oscillator: position, velocity, and acceleration. Remember these?

$$x(t) = A\cos(\omega t + \varphi) \qquad v(t) = -\omega A\sin(\omega t + \varphi) \qquad a(t) = -\omega^2 A\cos(\omega t + \varphi)$$

Yeah…I thought so.

For our LC circuit, the charge on the plates varies as: $q(t) = Q_{max}\cos\omega t$

And since current I is charge-per-time, i.e., the time-derivative of the charge function $q(t)$, that means the current in the circuit is given by: $I(t) = -\omega Q_{max}\sin\omega t$

…where ω is the angular frequency (Sound familiar?), given by: $\omega = \sqrt{\frac{1}{LC}}$

With mass-on-a-spring simple harmonic oscillators, the same quantity was: $\omega = \sqrt{\frac{k}{m}}$

And there you have it: the behavior of LC circuits. I am given to understand that you will NOT have to perform in-depth calculations with LC circuits on the AP Physics C Exam, but it IS fair game for them to ask you to know, conceptually, what is going on. (Thus, the above verbiage.) Just for fun (!), you ought to draw yourself a $q(t)$ graph, with an $I(t)$ graph right below it – like we did with $x(t)$, $v(t)$, and $a(t)$ for simple harmonic oscillators – and convince yourself that those graphs tell the exact same story as what we told with words above, in Steps 3-4-5, 3-4-5, 3-4-5…

It's been real. Best of luck to you in all your future endeavors.

Peace, out.

Mr. Bergmann