DC Circuit Analysis, Including RC Circuits

6.1 Sources of Electromotive Force, emf

An **emf source** (or **emf device**) is an energy-conversion device that changes chemical, mechanical, or some other form of energy into the electrical energy associated with charge separation. An emf source is able to maintain a separation of (+) and (–) charge, and thus maintain a potential difference ΔV, between its two terminals. For our purposes, the primary example of an emf device is the battery.

For the record, "emf" is a shorthand version of the historical term "electromotive force," which was coined in the early days of the study of electricity to refer to the "oomph" that moves charge through circuits. But emf is NOT a force; rather, it represents an amount of energy U per unit charge q and, back in Unit 3, we learned that energy-per-charge is…voltage. So today, emf is basically a ΔV, but it is (usually) a very specific ΔV; the term "emf" most commonly refers to the maximum potential difference ΔV that, say, a battery can impress on a circuit. Besides "emf," some synonyms for this maximum potential difference ΔV of which a battery is capable are "source emf," "ideal voltage," and "open-circuit voltage." The symbol of a battery's emf is $\mathcal{E}$, and the unit is, of course, volts (V).

An **ideal emf device** lacks internal resistance. Ideal emf devices thus maintain the same potential difference ΔV (i.e., the same voltage) across their terminals under ALL conditions. Specifically, for an ideal emf source, it makes no difference if the device is in operation (and thus has a current I flowing through it) or if the device isn't being used (and thus has NO current I flowing through it). Either way, the measured potential difference ΔV across the two terminals is the exact same, and it is equal to the emf $\mathcal{E}$.

Speaking of measured potential difference ΔV… ANY measured potential difference ΔV across two battery terminals is called a **terminal voltage** ΔV. So, in the case of an ideal emf device, the terminal voltage ΔV and the emf (or "source emf," or "ideal voltage," or "open-circuit voltage") are equal to each other.

Unlike an ideal emf source, a **real emf source** has internal resistance, indicated by a lowercase r. The internal resistance r in a real emf source causes the measured terminal voltage ΔV to depend on HOW MUCH current I is flowing through the source. The terminal voltage ΔV is maximized when ZERO current I is passing through the source, i.e., when the device isn't being used to push charge through a circuit. In that case, the ΔV across the terminals equals the emf $\mathcal{E}$, just like it always does in an ideal source.

However, when a real emf device is in use, there will be a current I passing through it. In such a case, the measured terminal voltage ΔV is somewhat reduced. For any real emf device, whether in operation or not, the terminal voltage ΔV can always be found using the equation:

$$\Delta V = \mathcal{E} - Ir$$

This equation supports our contention that the terminal voltage ΔV is a maximum (and is equal to the emf $\mathcal{E}$) when the current I is zero, as well as that the ΔV is reduced whenever a current I passes through the device during operation. It further reveals that, the greater the current I, the lower the measured potential difference ΔV between the terminals. The decrease in the terminal voltage ΔV is due to energy "losses" (i.e., "atoms-wiggling-faster energy") that necessarily arise whenever a current I flows (unless it is passing through a superconductor).

In this class, assume that all emf devices are ideal sources of emf, unless explicitly told otherwise. However, if a real emf source does arise, it is easily dealt with, in this way: Simply model the real emf source (which, you recall, has a nonzero resistance *r*) as an ideal emf source (having zero resistance) that is immediately adjacent to – i.e., is in series with – a simple resistor having the resistance *r*. The resistance *r* must then be accounted for when analyzing the circuit, e.g., determining equivalent resistance, and so on, but all you've really done is add one more resistor (having resistance *r*) to the circuit. No big deal.

6.2 Circuit Basics

This section goes through some concepts you met in first-year physics, as well as a few new ideas.

For starters, the purpose of circuits is to CONTROL electric current *I*. Lightning is a form of current, but we can't perform useful work with it because it isn't controlled. Circuits are designed and built to carefully control and distribute current; by doing so, much useful work can be accomplished.

Whenever a current *I* travels through a circuit, always flowing in the same direction and never reversing itself, we have a **direct current** (DC) circuit. A DC current need NOT always be of a constant magnitude – it may even cease completely from time to time – but as long as when it flows, it flows only in one direction, it is a DC current. On the other hand, if the current *I* were to go first in one direction, then stop and reverse itself, then stop and flow again in the original direction, etc. etc., we would have an **alternating current** (AC) circuit. In this class, we will deal only with DC circuits.

An **ammeter** is an instrument used to measure current *I*. (In your first-year class, you called current "flow rate," and you denoted it using "arrowtails.") The instrument's name is related to the unit for current *I* – amperes – and it is pronounced "AMM-meeter," NOT "AMMitter." (Sorry: Pet peeve, there…)

In order for an ammeter to quantify the current *I* flowing through a certain circuit element, it must be placed in SERIES with that element. This ensures that whatever current *I* flows through the element also flows through the ammeter, which measures it. Furthermore, if an ammeter had any appreciable resistance (which it doesn't), it would increase the resistance of that portion of the circuit, because the element and the ammeter are in series. The real, measured current *I* flowing in that part of the circuit would then be lower than it would be if the ammeter WEREN'T there, which defeats the purpose of having an ammeter in there: We need to know what the TRUE currents are in the circuit, i.e., what the currents are when there are only components and wires in the circuit. This is why ammeters must (and do!) have extremely LOW resistances, like wires do; we need to get our ammeter "in there" without the circuit knowing about it. From the circuit's point of view, an ammeter is just…an ordinary wire.

Because of their low resistances, ammeters are extremely susceptible to damage if they are improperly inserted into a circuit…such as if they are wired in parallel with other circuit elements. ("Curses!") If we place an ammeter in parallel with other circuit elements instead of in SERIES with them (like it should be), ALL the current will choose this path of least resistance through the ammeter instead of through the resistor(s), and we will get a very LARGE current through the ammeter. Given that it is large currents that damage electrical components…you can see why you must NEVER wire an ammeter in parallel.

A **voltmeter** measures potential difference ΔV. (In first-year physics, you called this ΔV "pressure difference," and it was a quantitative measure of the COLOR DIFFERENCE between two parts of a circuit.) The instrument's name comes from the unit for ΔV – volts – and is pronounced "VOLT-meeter," NOT "vol-TAHmitter." (Pet peeve #2, there…) In order to quantify the ΔV between two locations in a circuit, the voltmeter has to "touch" both of those locations. Voltmeters must therefore be wired in PARALLEL, around other circuit elements. Furthermore, a low resistance on the part of a voltmeter that is wired in parallel would cause current to flow THROUGH the voltmeter, BYPASSING whatever other elements are within the parallel region, rather than flowing through that region as the circuit was originally designed. Voltmeters therefore have extremely HIGH resistances, and they are much tougher to damage than are ammeters. (At least, using electrical means… ☺)

Here is a quick review of the equations needed to calculate the equivalent resistance of a sequence of resistors, depending on whether they are in series or in parallel. I refer to the series equation as the "add-'em-up" equation and the parallel equation as the "one-over" equation:

$$Resistors\ in\ series: \quad R_s = R_1 + R_2 + \cdots \qquad or \qquad R_s = \sum_i R_i$$

$$Resistors\ in\ parallel: \quad \frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots \qquad or \qquad \frac{1}{R_p} = \sum_i \frac{1}{R_i}$$

Because they are all in a row, one after the other, resistors in series all share a common current *I*. Because they are all bridges between two specific regions of a circuit, with each region having its own potential *V*, resistors in parallel all share a common potential difference *ΔV*. If you can imagine a ladder…One rail is at one potential *V* (in terms of CASTLE, it has "some" color); the other rail is at a different potential *V* (it has "some different" color). Do you see how every single rung of the ladder – each of which represents a resistor in parallel – therefore has the same *ΔV* (i.e., the same "color difference") across it?

6.3 Kirchhoff's Rules for Circuit Analysis

German physicist Robert Kirchhoff (variously pronounced "KEERK-off" or "KERK-off") is credited with formulating two rules useful in analyzing electric circuits. These rules apply to all circuits: not only AC and DC circuits, but also circuits that contain any combination of emf sources, resistors, capacitors, inductors, and any other type of electrical element. In this unit, we will deal with DC circuits that either have emf-sources-and-resistors-only or emf-sources-and-resistors-and-capacitors-only.

Kirchhoff's Junction Rule (or **Node Rule**) says that the sum of the currents I entering any junction must equal the sum of the currents I exiting that junction. This rule is a consequence of the law of conservation of charge. You were already exposed to this rule in the electricity studies in your first-year physics class.

Kirchhoff's Loop Rule says that the algebraic sum of the CHANGES in potential ΔV encountered in a complete traversal of any loop of a circuit equals zero. Imagine some arbitrary point in a circuit: It has some potential V; in CASTLE terms, it has "some color." The potential DIFFERENCE ΔV between this point and itself is obviously…zero. So if we start at that point, travel around some loop in the circuit, and end up back at that point, the amount by which our potential will have changed by must be…zero. In equation form, the Loop Rule might look like:

$$\sum_{loop} \Delta V = 0$$

If the algebraic sum of all the ΔVs around the loop is zero, then that necessarily means that some of the ΔVs we encounter will be (+), while other ΔVs will be (–).

The Loop Rule, too, you have already had experience with, although it wasn't as explicitly taught in your first-year class as was the Junction Rule. The Loop Rule is a consequence of the law of conservation of energy.

To use the Loop Rule, you have to start at some point on a circuit and make one complete loop, ending at the same place you started. It won't matter the exact location where you start on the path, nor will it matter if you travel clockwise or counterclockwise around the loop. As long as you correctly follow the rules of circuit analysis, you will obtain the correct results.

As you traverse the circuit elements around your chosen loop, you will need to add or subtract, as appropriate, the potential difference ΔV associated with every circuit element you meet along the way. If the circuit element is, say, a battery having an emf $\mathcal{E}$, then this potential difference ΔV will be either $+\mathcal{E}$ or $-\mathcal{E}$, depending on how the direction in which you are going around the loop compares to the orientation of the battery. If the circuit element is a resistor R, then this potential difference ΔV will be (by Ohm's law) either $-IR$ or $+IR$, depending on how the direction you are going around the loop compares to the assumed direction of the (usually) unknown current through that resistor. This is difficult to understand without context, but – fear not – we will go over examples during class. For the sake of completeness, here are the specific sign conventions for the various ΔVs that were hinted at above:

1. When an emf source $\mathcal{E}$ is traversed in the direction OF the emf (i.e., – to +), the ΔV across the source equals $+\mathcal{E}$. When traversed OPPOSITE TO the emf (i.e., + to –), the ΔV across the source equals $-\mathcal{E}$.
2. When a resistor R is traversed in a direction going WITH the assumed current I, the ΔV across the resistor equals $-IR$. When traversed in a direction OPPOSITE the assumed current, the ΔV equals $+IR$.
3. When a capacitor C is traversed in the direction that the $\boldsymbol{E}$ field between its plates points (i.e., from + to –), the ΔV across the capacitor equals $-{}^{Q}/_{C}$. When a capacitor is traversed OPPOSITE TO the $\boldsymbol{E}$ field between its plates (i.e., – to +), the ΔV across the capacitor equals $+{}^{Q}/_{C}$.

The result of you traversing the loop-of-your-choice in the direction-of-your-choice will be an equation that corresponds to that specific loop and that specific direction. For some circuits we will examine, you will need several different loops in order to fully analyze the circuit; specifically, you will need as many unique loops (and thus equations) as you have unknown quantities.

Again, all of this is hard to grasp without seeing examples, but these will be done in class so that you will understand. But here are a few things to keep in mind, once we get into analyzing DC circuits:

1. Assume as few independent, unknown currents as possible.
2. Assume each unknown current *I* to go in some direction (clockwise (CW) or counter-clockwise (CCW)). It won't matter which way, but draw an arrow on each *I* to show the direction you assumed. At the end, if a current *I* comes out (+), you assumed the correct direction. If a current *I* comes out (–), then you assumed wrongly; no big deal. That current *I* actually goes the opposite way.
3. Select as many independent loops as you have unknown currents. Keep in mind that each of your loops must traverse some circuit element that is NOT traversed by ANY other loop.
4. For each loop, choose where you will start. Put a dot or a star or something at that point; you will stop at that same point, after going around the loop.
5. Once you have all of your loop equations written, you will need to solve them by setting up a system of equations. YES! (good heavens, YES!) you will be able to simply plug that system of equations into your graphing calculator to obtain the answers to your unknowns. (I'm not THAT much of a rat-bastard…)

In all, this process isn't as bad as it sounds. You'll see, once we show you some examples. You WILL get it.

6.4 RC Circuits

Circuits with a resistor and a capacitor are called **RC circuits**. In this class, each such circuit will also include an emf source $\mathcal{E}$.

You should have learned in earlier physics studies that it takes time to move charge onto and off the plates of a capacitor, particularly if there are resistance elements in the circuit. In other words, the potential difference ΔV across the plates of a capacitor does not change instantaneously. So we shouldn't be surprised to learn that capacitors introduce time-dependent behavior into electric circuits.

The product of the resistance R and the capacitance C is called the **capacitative time constant** τ (or just **time constant**):

$$\tau = RC$$

The time constant τ deals with the rate at which a capacitor charges and discharges in an RC circuit. The larger the time constant τ, the more time required for charging and discharging. The SI unit for the time constant τ is the same as the SI unit for time: seconds (s). Based on the equation, you might think the unit for τ should be the ohm-farad (Ω-F). Well...it is! One Ω-F is better known as one second. It's a fun exercise to try using a couple of basic physics equations to prove to yourself that this is so. Try it! Anyway...

It is important that you understand conceptually how a capacitor behaves in a circuit. While equations will be provided that will enable you to calculate various quantities in RC circuits at precise times, you must have an intuitive feel for the circuit's behavior prior to diving into the equations. In Sections 6.5 and 6.6, we will walk through what happens, conceptually, when charging and discharging a capacitor.

6.5 Capacitor Charging in RC Circuits

1. In an RC circuit, a capacitor acts like a simple wire at the first instant of connection; it acts likes a perfect insulator after significant time has passed. (You were actually exposed to this behavior of RC circuits many, many times in your first-year physics class.) This means that, in an RC circuit, the current I is largest initially and eventually decreases to zero. Visually, a "charging" graph (as opposed

to a "discharging" graph, to be discussed in Section 6.6) of current I vs. time t would begin NOT at zero, descend steeply at first, and then become gradually less steep until it finally tapers off to a slope of zero, at which point the current I (which is graphed along the y-axis) will also have a value of zero.

2. When charging a capacitor in an RC circuit, the net charge q on the plates begins at zero, increases rapidly at first, then continues increasing, but more and more slowly until the amount of net charge q on the plates ceases to change. Such a graph of charge q vs. time t would begin at zero, rise steeply at first, and then become gradually less steep until it finally tapers off to a slope of zero, at which point the charge q is a maximum. Thus, for charging, a q vs. t graph is essentially an I vs. t graph that has been flipped upside down, with the vertical scale adjusted and the units on the vertical axis changed.

3. During charging, the potential difference ΔV between the capacitor plates follows the same time-dependent trend that the charge q does. Like the q vs. t graph mentioned above, a "charging" graph of potential difference ΔV vs. time t begins at zero, rises steeply at first, and then becomes gradually less steep before tapering off to a slope of zero, at which point the ΔV between plates is maximized.

A key idea you must gather from paragraphs 2 and 3 above is that the net charge q on the plates and the potential difference ΔV across the plates are always directly proportional, and the constant of proportionality between them is the capacitance C of the capacitor. Perhaps looking at the familiar equation for capacitance in a new way will help you understand how the graphs mentioned in paragraphs 2 and 3 relate to one another. Previously, we saw:

$$C = \frac{Q}{\Delta V}$$

Now, we could write:

$$C = \frac{q(t)}{\Delta V(t)} \qquad OR \qquad C = \frac{Q(t)}{\Delta V(t)}$$

Unlike charge q, current I, and voltage ΔV, the capacitance C is a constant, for a given capacitor. The equations at the bottom of the previous page hint at this time-invariant nature of the capacitance C.

The equations that precisely show the trends during capacitor charging are essentially the equations of the lines on the graphs described above in paragraphs 1, 2, and 3. Here they are, with a short preamble:

If a capacitor of capacitance C is charged with a battery of emf $\mathcal{E}$ through a circuit of resistance R, the charge q on the capacitor plates, the current I in the circuit, and the potential difference ΔV between the plates (not across the battery terminals!) vary in time, according to:

$$q(t) = C\varepsilon\,(1 - e^{-t/\tau}) \qquad I(t) = \frac{\varepsilon}{R}\,e^{-t/\tau} \qquad \Delta V(t) = \varepsilon\,(1 - e^{-t/\tau})$$

where the instant of connection is assumed to occur at $t = 0$.

Based on our remembering that "anything-to-the-zero-power equals one," we note a few things from the above equations. At $t = 0$...

a. the charge q on the plates is zero, i.e., $q(0)$ = zero

b. the current I is maximized and has the magnitude $\mathcal{E}$/R, i.e., $I(0) = \mathcal{E}/R$

c. the potential difference ΔV across the capacitor plates is zero, i.e., $\Delta V(0)$ = zero

We also see that, as t gets very large, relative to τ...

a. the charge q on the plates approaches its maximum value of C$\mathcal{E}$, i.e., $q(\infty) = C\mathcal{E}$

b. the current I approaches its minimum value of zero, i.e., $I(\infty)$ = zero

c. the potential difference ΔV across the plates approaches its max value of $\mathcal{E}$, i.e., $\Delta V(\infty) = \mathcal{E}$

6.6 Capacitor Discharging in RC Circuits

When the emf source (i.e., the battery) is removed from an RC circuit and the capacitor is discharged:

1. Again, the magnitude of the current I is largest initially, decreases rapidly at first, then decreases more gradually until it reaches zero, at which point the capacitor is fully discharged. A "discharging" graph of I vs. t would begin NOT at zero, change steeply at first, then change gradually until the slope reaches zero, with the curve itself approaching an I value of zero. Sometimes, for a "discharging" I vs. t graph, the initial current I is graphed as a maximum NEGATIVE value, with the curve sloping UPWARD – rapidly at first, less rapidly later on – with all currents being (–), until the curve rises to an ending current I of zero. The reason for the (–) currents is that, during discharge, the current DIRECTION is opposite to that during charging. NOT all "discharging" I vs. t graphs employ the (–) current idea, but if you understand what is going on here, you can deal easily with any "discharging" I vs. t graph you're given.

2. During discharge, the amount of charge q on the plates decreases rapidly at first, then more and more slowly, until the amount of charge q on each plate is zero. A "discharging" graph of q vs. t – you might be able to guess by now – slopes downward rapidly at first, then eventually merges smoothly into the time axis, where q becomes equal to zero. Unlike a "discharging" I vs. t graph, where the initial value might be (+) or it might be (–), we need not worry about whether q will be (–) or not: It won't be; it will be (+).

3. During discharge, the potential difference ΔV between the plates mirrors the trend described above for charge q: starting NONzero and (+), decreasing rapidly at first, then more gradually, until both the slope and the value of ΔV reach zero.

Like the capacitor-charging equations, the equations for discharging are also exponential in nature. Here, t has been redefined such that $t = 0$ at the moment discharging begins.

$$q(t) = C\varepsilon\, e^{-t/\tau} \qquad I(t) = -\frac{\varepsilon}{R}\, e^{-t/\tau} \qquad \Delta V(t) = \varepsilon\, e^{-t/\tau}$$

The (−) sign in the above equation for current I supports the commentary given earlier: It indicates nothing more than that the direction of the current I is opposite to what it was during charging.

Again, the equations above verify what we should already understand conceptually about a circuit with a discharging capacitor. At the moment discharging begins (i.e., at $t = 0$):

a. the charge q on the plates is at its maximum value of $C\mathcal{E}$, i.e., $q(0) = C\mathcal{E}$

b. the current I is maximized and has the magnitude $\mathcal{E}/R$, i.e., $I(0) = \mathcal{E}/R$ or $-\mathcal{E}/R$

c. the potential difference ΔV across the plates is at its maximum value of $\mathcal{E}$, i.e., $\Delta V(0) = \mathcal{E}$

At a much later time (i.e., as t becomes very large):

a. the charge q on the plates approaches zero, i.e., $q(\infty)$ = zero

b. the current I approaches its minimum value of zero, i.e., $I(\infty)$ = zero

c. the potential difference ΔV across the plates approaches zero, i.e., $\Delta V(\infty)$ = zero

The capacitor is then fully discharged.

6.7 The Significance of the Amount of Time t That's Equal to One Time Constant τ

You recall thinking about all of the graphs described in Sections 6.5 and 6.6: they all start somewhere, change rapidly at first, then change more gradually, eventually the slope goes to zero, blah-blah-blah. Here is a crucial idea for you to understand: Starting at $t = 0$, after a time t that is equal to one time constant τ has elapsed, 63% of the total change that is going to happen…has happened. In other words, after the start (of either charging or discharging), when t equals whatever-the-value-of-τ-is, every quantity is already 63% of the way to wherever it's going to finally end up. Specifically:

During charging, when $t = \tau$:

(a) the charge q on the plates is 63% of the way UP to its maximal, ending value

(b) the plate voltage ΔV is 63% of the way UP to its maximal, ending value

(c) the current I is 63% DOWN from its max, initial value (i.e., it's at only 37% of its initial max value)

During discharge, when $t = \tau$: (remember, "initial" now means "the moment discharging begins")

(a) the charge q is 63% DOWN from its max, initial value (i.e., it's at only 37% of its initial max value)

(b) the voltage ΔV is 63% DOWN from its max, initial value (i.e., it's at only 37% of its initial max value)

(c) the current I is 63% DOWN from its max, initial value (i.e., it's at only 37% of its initial max value)

The significance of the 63% is this: Every time-dependent equation in Sections 6.5 and 6.6 had either…

$$1 - e^{-t/\tau} \quad or \quad e^{-t/\tau}$$

…in it. When $t = \tau$, the "e" parts above become e^{-1}, which is about 0.37. So $1 - e^{-1} = 0.63$, and $e^{-1} = 0.37$. Obviously, 0.63 and 0.37 are different numbers ("duh…!") but, in terms of the physics of RC circuits, the commonality that they both conform to is: When t equals whatever-the-value-of-τ-is, every quantity is already 63% of the way to wherever it's going to finally end up, either 63% of the way UP to a maximum value, or 63% of the way DOWN to zero. And – when a SECOND amount-of-time τ has elapsed – every quantity has changed by an additional 63%-of-whatever-is-left-to-go-after-the-first-τ-has-done-its-thing.