<u>Capacitance</u>

4.1 Capacitors and Electric Potential Energy U

A **<u>capacitor</u>** consists of two conductors separated by either an insulating material or a vacuum. There are several types of capacitors, but the one we'll refer to most often – the **<u>parallel-plate capacitor</u>** – can be thought of as an "insulator sandwich, with conductor bread."

A capacitor stores electric potential energy U when its two conductors are given equal but opposite charges Q. Energy U is stored in a capacitor because the charging process requires work to be done when moving charges from one conductor to the other. When the charge that is moved is taken to be (+) (which is the universal convention) the plate that gains charge (the + plate) ends up with a higher potential V, while the plate that loses charge (the – plate) ends up with a lower potential V. (Perhaps you might remember learning this already, in an earlier physics class, given that another term for electric potential V is "electric pressure.") This movement of charge Q and the resultant storing of potential energy U is somewhat analogous to potential energy U being stored when a mass m is moved in a gravitational field $\boldsymbol{g}$, from a lower elevation to a higher one. It should be noted that, when charging a capacitor, charges are simply moved off of one plate and added to the other; therefore, the total net charge contained in a capacitor, strictly speaking, is always zero.

One useful property of a capacitor is that it can store energy that is built up over some longer period of time and then be made to discharge that energy in a much shorter period of time. That differing rate of energy transfer is possible only when the resistance of the discharging circuit is much lower than that of the charging circuit. Two examples are when capacitors are used to deliver a powerful, short burst of energy during a camera flash or a portable defibrillator shock.

The transferring of charge Q from one plate to the other results in the generation of a uniform electric field $\boldsymbol{E}$ between the plates. (Recall that a uniform field can be visualized as a series of evenly-spaced, parallel arrows.) The direction of the uniform $\boldsymbol{E}$ field points from the (+) plate (which has the higher potential V) to the (−) plate, which has the lower potential V. From the equation given at the end of Section 3.4, namely:

$$\Delta V = -\int_A^B \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -E\int_A^B ds = -Es$$

...we see that, for a uniform $\boldsymbol{E}$ field between two parallel plates separated by a distance s (or d, or x), the electric field $\boldsymbol{E}$ that exists between the plates has the magnitude:

$$E = \frac{\Delta V}{d}$$

The potential energy U that is stored in a capacitor is stored in the electric field $\boldsymbol{E}$ that exists between its plates...just like gravitational potential energy U_g is actually stored IN the gravitational field $\boldsymbol{g}$ whenever a mass is raised to a new and higher elevation.

The electric potential energy U of a charged capacitor is numerically equal to the work W required to move the charges Q when we charged it. (As an aside, recall that work and energy have the same unit (i.e., joules J). And furthermore, it is no coincidence that when work is done, energies change OR when energies change, work is done.) Anyway, it can be shown that the potential energy U can be calculated using any two of the following three variables: the net charge on one plate Q, the capacitance of the capacitor C, and the potential difference between the plates ΔV:

$$U = \frac{Q^2}{2C} = \frac{1}{2}\, Q\Delta V = \frac{1}{2}\, C(\Delta V)^2$$

4.2 Capacitance C

Capacitance C is a measure of a capacitor's ability to hold charge. Capacitance C is defined as the ratio of the amount of charge Q on one of a capacitor's conductors (i.e., one of its plates) divided by the potential difference ΔV between the conductors:

$$C = \frac{Q}{\Delta V}$$

As you can see from the equation, the greater the capacitance C, the more charge Q can be stored on the plates for a given potential difference ΔV between plates.

Capacitance C depends only on the geometry of the capacitor and the materials used in its construction. Said another way: The capacitance of a capacitor does not depend on any external source of charge Q or any applied potential difference ΔV. The capacitance of a capacitor is a physical property of that capacitor. ("The capacitance of a capacitor...is what it is.") As mentioned already, when some amount of charge Q is transferred from one plate to the other, a potential difference ΔV (and an electric field $\boldsymbol{E}$) appears between the plates. If more charge Q is moved, the potential difference ΔV and electric field $\boldsymbol{E}$ both increase as well. No matter how much charge Q is transferred, the potential difference ΔV changes proportionally; the constant of proportionality is the capacitance C. This trend, too, is seen in the equation above.

In a circuit, the potential difference ΔV between the two plates can typically increase until it matches the potential difference ΔV across the battery terminals. At that point, the battery cannot push any more charge from one plate to the other; charge Q stops increasing, the ΔV between the plates stops increasing, and the $\boldsymbol{E}$ field between the plates stops increasing.

For a parallel-plate capacitor with plates separated by a distance *d*, with a vacuum (or air) in the intervening space, and with each plate having the area *A*, the capacitance *C* can be found using:

$$C = \frac{\epsilon_o A}{d}$$

ε_o (the permittivity of free space) used in the above equation is the by-now-familiar...

$$\epsilon_o = 8.85 \; x \; 10^{-12} \; \frac{C^2}{N \cdot m^2}$$

With reference to the above equation...Increasing the plate area *A* increases the capacitance *C* because, by making the surface bigger, a greater quantity of charge *Q* is more easily moved from one plate to the other. Decreasing the plate separation *d* increases the capacitance *C* because, due to the smaller distance between the plates, the opposing charges on each plate can hold each other more tightly in place than they could before. This extra holding-power allows more charge *Q* to be moved for a given potential difference *ΔV*, resulting in a higher capacitance *C*.

4.3 The Dielectric Constant κ

A physical material insulator inserted between a capacitor's two plates is called a **dielectric**. Examples of dielectrics include rubber, glass, waxed paper, or any of various plastics. A dielectric always INCREASES the capacitance of a capacitor. In order that the dielectric has its precisely anticipated effect, it is very important that a dielectric material fill the entirety of the space between the plates, with no air gaps.

When a capacitor having a dielectric is charged, the electric field ***E*** between the plates produces aligned **dipoles** in the molecules of the dielectric. This means that, instead of the electron clouds of those molecules being centered on each nucleus (as would normally be the case), the electron cloud for each molecule is shifted to one side, giving that side of the molecule a tiny, net (–) charge and the opposite side of that molecule a tiny, net (+) charge. It goes without saying that the negatively-charged electron clouds shift toward the (+) plate, i.e., the one acquiring the higher potential V. The molecular dipoles effectively DECREASE the magnitude of the electric field $\boldsymbol{E}$ within the dielectric material because the internal $\boldsymbol{E}$ field that appears within each molecule due to the shifted electron-cloud is OPPOSITE that of the larger $\boldsymbol{E}$ field that exists between the capacitor's plates. Let's give a conceptual example, and perhaps it will help…

Imagine yourself in a classroom. The ceiling of the classroom represents one plate of a capacitor; the floor represents the other plate. So you see that, since you're IN the room, you are actually IN the space between the plates. We now charge this classroom-capacitor, such that the ceiling experiences a net GAIN of conventional charge ($+Q$) and the floor experiences a net LOSS of conventional charge ($-Q$). Thus, the ceiling is now Positively charged and has a higher potential V, while the floor is Negatively charged and has a lower potential V. Imagine a bunch of copies of the uppercase letter 'P' (for Positive) hanging on the ceiling, and a bunch of copies of the uppercase letter 'N' (for Negative) covering the floor. The electric field $\boldsymbol{E}$ throughout the classroom is uniform – if we ignore the slight **fringing** that occurs at the edges – so we need to visualize a bunch of straight, parallel, evenly-spaced, downward-pointing P-to-N arrows all throughout the room. These arrows indicate the $\boldsymbol{E}$ field between the P-ceiling plate and the N-floor plate. As an observer sitting in the classroom-capacitor, you can (somehow) sense the strength of this $\boldsymbol{E}$ field.

Now, we are going to fill the room with a dielectric. We bring in as many inflated balloons as are needed to fill the available space. (Don't worry: This is just a thought experiment; you can stay in the room without harm, so don't feel like you have to morph into a spirit animal or a berserker and start madly popping balloons in a desperate attempt to avoid asphyxiation.) The balloons represent molecules of the dielectric, with the exterior of each balloon being the electron cloud. The nuclei of the molecules' atoms are buried inside the dielectric-balloon molecules, where we can just barely discern them, because the balloon material is very light in color and somewhat transparent. Somehow, the balloons don't disturb the many straight, parallel, evenly-spaced downward-pointing, ***E***-field-representing…P-to-N arrows.

Okay, let's observe these balloon-dielectric molecules. If you look closely, you will see that one lowercase 'p' (positive) and one lowercase 'n' (negative) has appeared on each balloon. Furthermore, on EVERY SINGLE balloon-dielectric molecule, the 'p' has appeared on the BOTTOM of the balloon, closer to the N-floor, while the 'n' has appeared on the TOP of the balloon, closer to the P-ceiling. This appearance of 'p' and 'n' on every balloon-molecule is a consequence of the slight shifting of each molecule's electron cloud, due to the ***E*** field between the P-ceiling plate and the N-floor plate. Within THIS classroom-capacitor, these electron clouds are at a slightly HIGHER elevation than they would be in an uncharged classroom-capacitor because the (–) electron cloud experiences a force OPPOSITE the ***E*** field which, you recall, points from the P-ceiling to the N-floor. Thus, each electron cloud is forced slightly UPWARD.

Almost done. Focus in on a single dielectric-balloon molecule. Notice through the semi-transparent balloon membrane that there is a tiny arrow INSIDE the balloon-dielectric molecule that points UPWARD, originating on the molecule's 'p' and terminating on its 'n'. In fact, you notice now that ALL of the balloons have these upward-pointing arrows; thousands of them, all pointing up, while the larger P-ceiling-to-N-floor arrows continue to point downward. The smaller arrows inside each balloon represent the tiny, localized ***E*** fields generated WITHIN each balloon's **dipole**, each of which was created when the electron clouds shifted, due to the influence of the P-ceiling-to-N-floor ***E*** field. The NET ***E*** field that now exists within the classroom-capacitor is the superposition of the P-ceiling-to-N-floor ***E*** field AND the thousands of tiny (and opposite-pointing) balloon-dielectric-molecule ***E*** fields. And here's the kicker: As an observer sitting in the classroom-capacitor, you now perceive that the strength of this new, superposed ***E*** field is WEAKER than the original P-ceiling-to-N-floor ***E*** field. The ***E*** field between the plates has been REDUCED. And that's the deal.

It turns out that when a dielectric is inserted between the plates of a capacitor, not only is the ***E*** field reduced, but also the capacitance C increases by a dimensionless factor κ, called the **dielectric constant**. Thus, whenever a dielectric material is present, a parallel-plate capacitor has a capacitance given by:

$$C = \frac{k\, \epsilon_o A}{d}$$

Based on how the above equation differs from the one at the end of Section 4.2 – and given that a dielectric always increases the capacitance – it is easily seen that the dielectric constant κ is always greater than 1. If the region between plates is filled with air or has a vacuum, we consider $\kappa = 1$.

Another way of describing the dielectric constant κ is to say that it is the ratio of the applied ***E*** field (between the two plates) to the reduced ***E*** field (within the dielectric material), that is:

$$k = \frac{E_{applied}}{E_{dielectric}}$$

There are some interesting scenarios that arise with regard to Q, ΔV, C, and ***E***, depending on WHEN the dielectric material makes an appearance in the capacitor...

Was the dielectric material in the capacitor from the start?

If so, what happens to Q, ΔV, C, and ***E*** *if, after charging the capacitor, we remove the dielectric?*

Or what if the capacitor doesn't have a dielectric, then we charge it, and THEN we insert a dielectric?

What if the dielectric fills only part of the space between the plates, rather than all the space?

These are topics we will discuss in class, but let's consider two cases here, to give you a taste of the logic.

CASE 1: *Inserting a dielectric into a charged capacitor that is NOT connected to a battery.*

Suppose we have a circuit with a battery and a capacitor with NO dielectric between the plates.

A. Let us charge the capacitor with a ΔV due to the battery. We now have a certain charge Q on each plate (+Q on one plate and –Q on the other, obviously), a potential difference ΔV between plates that equals the ΔV across the battery terminals, and an electric field ***E*** between plates. The ***E*** field points from the top (+) plate (i.e., the one having +Q) toward the bottom (–) plate (i.e., the one having –Q).

B. Remove the battery from the circuit. The capacitor remains charged.

C. At this point, insert the dielectric, completely filling the space between plates. The charge Q on each plate is unaffected by this change, as there is nowhere for the charge to go.

D. However, the aligned dipoles of the dielectric molecules produce their own ***E*** field that opposes the original one between the plates. The result is a NEW effective ***E*** field between the charged plates; this new ***E*** field is smaller than the original one.

E. But, as discussed previously, the ***E*** field between two parallel plates is related to the potential difference ΔV and the separation d between them by:

$$E = \frac{\Delta V}{d}$$

So we can see that, if ***E*** is smaller and d remained the same, then ΔV must have gotten smaller. That is, by inserting the dielectric into the capacitor – without a battery to maintain the existing ΔV – the potential difference ΔV between the plates has decreased.

F. And now, by the equation...

$$C = \frac{Q}{\Delta V}$$

...we see that, for the situation we have just described – for which Q remains the same and ΔV decreases – we conclude that the capacitance C has increased.

CASE 2: *Inserting a dielectric into a charged capacitor that REMAINS connected to a battery.*

Again, we have a circuit with a battery and a capacitor with NO dielectric between the plates.

A. We again charge the capacitor with the battery. Again, we obtain a charge Q on each plate (one $+Q$ and one $-Q$), a potential difference ΔV between plates that equals the ΔV of the battery, and an electric field $\boldsymbol{E}$ between the plates.

B. This time leaving the battery in the circuit, we insert the dielectric between plates.

C. As before, the aligned dipoles of the dielectric molecules create an $\boldsymbol{E}$ field that partially cancels the original $\boldsymbol{E}$ field. The $\boldsymbol{E}$ field magnitude thus (momentarily) drops. By the equation...

$$E = \frac{\Delta V}{d}$$

...the potential difference ΔV between the plates also (momentarily) drops.

D. But the battery is still connected, and so it won't allow the ΔV to drop for long. As soon as the ΔV starts to drop, the battery immediately starts moving MORE charge Q, and continues doing so until the original ΔV is re-established.

E. Thus, we have a situation where the charge Q on the plates has increased while the potential difference ΔV across the plates has remained the same. And by the same equation as before...

$$C = \frac{Q}{\Delta V}$$

...we conclude that, here again, the insertion of a dielectric causes an increase in the capacitance C.

4.4 Circuit Properties of Multiple Capacitors

When multiple capacitors are present in a circuit we want to analyze, it is often useful to simplify the circuit and thus obtain specific, key quantities. Then, with those key quantities safely in hand, the simplified circuit can be "re-complicated" to obtain any quantity desired with regard to the various, real circuit elements. You might recall simplifying circuits in a previous physics course by finding the **equivalent resistance** R_{eq} of multiple resistors. For multiple resistors, the equivalent resistance R_{eq} is given by these formulas:

$$\textit{Resistors in series}: \quad R_s = R_1 + R_2 + \cdots \qquad or \qquad R_s = \sum_i R_i$$

$$\textit{Resistors in parallel}: \quad \frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots \qquad or \qquad \frac{1}{R_p} = \sum_i \frac{1}{R_i}$$

As you will see below, the series/parallel equations for resistors are essentially flip-flopped for capacitors. For multiple capacitors, the **equivalent capacitance** C_{eq} is given by the formulas shown:

$$\textit{Capacitors in parallel}: \quad C_p = C_1 + C_2 + \cdots \qquad or \qquad C_p = \sum_i C_i$$

$$\textit{Capacitors in series}: \quad \frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots \qquad or \qquad \frac{1}{C_s} = \sum_i \frac{1}{C_i}$$

From the basic equation for a parallel-plate capacitor, namely:

$$C = \frac{\epsilon_o A}{d}$$

...the parallel- and series equations above make intuitive sense, in the following two ways:

1. From the equation for C_p (the "add 'em up" equation), it is obvious that the equivalent capacitance C_p is always larger than any one of the individual capacitances. This is because connecting capacitors in

parallel is equivalent to increasing the plate area A which, as we can see by the equation above, results in an increased capacitance C.

2. From the equation for C_s (the "one-over" equation), the equivalent capacitance C_s is always smaller than any one of the individual capacitances. This is because connecting capacitors in series is equivalent to increasing the plate separation d which, as we can see by the equation above, results in a decreased capacitance C.

The next two points are also important to know, with regard to circuits containing multiple capacitors:

1. The potential difference ΔV is the same across two or more capacitors in parallel. This is because the connected top plates of the multiple capacitors form an equipotential surface, as do the connected bottom plates. Thus, all of the connected top plates have one particular potential V, and all of the connected bottom plates have a different particular potential V…which means that, across each pair of plates, the ΔV must be the same.

2. The charge Q on each plate is the same for two or more capacitors in series. This is a consequence of the law of conservation of charge, as well as of the fact that the insulating material between each capacitor's plates prevents any net charge flow.

4.5 Modification of Gauss's Law for Capacitors With a Dielectric

Recall Gauss's law, first introduced in Section 2.6:

$$\varphi_E = \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{q_{in}}{\epsilon_o}$$

If using Gauss's law on a capacitor having a dielectric, the law must be modified as follows:

$$\varphi_E = k \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{q_{in}}{\epsilon_o}$$