Electric Potential

3.1 Electric Potential Energy U

The work it would take to assemble a particular charge distribution (with all charges brought in from a distance of infinity) is stored as the **electrostatic potential energy** *U* of the distribution. This potential energy *U* resides in the electric field ***E***. If the ***E*** field is somehow altered, then the energy *U* is altered, too; energy has been either added to or removed from the field.

The potential energy *U* associated with two point charges that are separated by a distance *r* is given by:

$$U = k\,\frac{q_1\,q_2}{r} = \frac{1}{4\pi\epsilon_o}\,\frac{q_1\,q_2}{r}$$

Once again, this energy *U* represents the work done by an external agent when the charges q_1 and q_2 are brought in from an infinite separation to the separation *r*.

The potential energy *U* of a collection of point charges (say, *n* of them, where *n* > 1) is compatible with the principle of superposition. That is, the total *U* is found by summing the results of the above equation for each unique pair of the *n* charges. For *n* charges, the above equation will have to be used *x* times, where:

$$x = \frac{n\,(n-1)}{2}$$

The process described above is analogous to that used to find the absolute gravitational potential energy U_g associated with a collection of masses. The equation was:

$$U_g = -G\,\frac{m_1\,m_2}{r}$$

and we used it x times for a set of n masses, just like what we need to do to get U for a set of n charges. Note how the format of the two equations is identical: a constant, multiplied by two other quantities (of the same type as each other), and divided by the distance r between them.

Although the processes described above are identical, be aware that U_g for each result is always negative; when summed up, they simply give a more negative result. Electric potential energy U, on the other hand, can be (+) or (−). It is (+) with like charges; it is (−) with unlike charges. The final U must account for this so, when you "add," it is possible that you'll have to do some subtracting, and your final result could be either (+) or (−).

3.2 Electric Potential V

The **electric potential** (or just **potential**) V is an electrical quantity (i.e., a number, with a unit) that is associated with some particular location, perhaps in space or, say, on a capacitor plate. The potential V at some location – which has the unit volts (V) – is the electric potential energy U that a charge q that is AT that location would have…divided by the amount of charge q, i.e.:

$$V = \frac{U}{q}$$

The important concept here is that the electric potential V at a particular location applies to that location whether or not we actually HAVE a charge there. And if we do...then it doesn't matter how many coulombs of charge it is: The potential V at some point has the same value for any charge (or no charge at all) that's AT that point. And "the point" we're referring to here could mean a point in space, or perhaps a point on a material substance (like a circuit element, such as a capacitor plate). In either case, the point could be described as having a certain electric potential V, and that potential V provides energy U to a charge q that shows up there, by the equation above, i.e., $U = q\,V$.

Incidentally, two equations that will get you FAR in electricity are: $\boldsymbol{F} = q\,\boldsymbol{E}$ and $U = q\,V$

One analogy is that the electric potential *V* at some point in space is like the altitude at some location above sea level; there's a number associated with it, and it makes no difference if there is any charge or mass at that location: the number still applies and has meaning. Another analogy is that the electric potential *V* at some point in space is like a mile marker on the interstate; there's a number associated with it, and it makes no difference if there is any charge or mass at that location: the number still applies and has meaning. Again, in each case (potential *V*, altitude, or a mile marker), the numbers vary with location, and there doesn't always have to BE anything at those locations for those numbers be useful and to allow us to calculate other quantities.

Like energy *U*, electric potential *V* is a scalar quantity; there is no directionality to either quantity, although both quantities are allowed to be (+) or (−). The scalar nature of potential *V* makes it somewhat easier to deal with, compared to vector quantities such as the electric field ***E***, for which one must not only consider the magnitude, but also the direction. For potential energy *U* and potential *V*, only magnitudes are relevant. (You might not realize it now, but this is an extremely pleasant fact.)

Just like the potential energy *U* is assumed to be zero at a distance of infinity, so too is the potential *V* assumed to be zero at infinity. Technically, when we speak of the potential *V* at some particular location, we truthfully mean the potential difference ΔV between that point and infinity (where *V* is, again, zero).

In other cases, we will simply define the potential *V* to be zero at some arbitrary location; this "zero" of potential *V* is then referred to as **ground**. As we will see, the important thing usually ends up being the electric potential DIFFERENCE ΔV between two locations.

By combining two equations given previously, namely…

$$U = k\,\frac{q_1\,q_2}{r} \qquad and \qquad V = \frac{U}{q} \qquad (\,i.e., \quad U = qV\ )$$

it is easily shown that the electric potential V a distance r away from a point charge q is given by:

$$V = k\frac{q}{r} \;=\; \frac{1}{4\pi\epsilon_o}\,\frac{q}{r}$$

where, as before,

$$k = 9\ x\ 10^9\ \frac{N\cdot m^2}{C^2} \qquad and \qquad \epsilon_o = 8.85\ x\ 10^{-12}\ \frac{C^2}{N\cdot m^2}$$

Once again, while there is no directionality to V, the equation DOES hold true with regard to sign; when using the equation, you must take the sign into account. If q is (+), V is (+); if q is (–), V is (–). On a side note, we can see that this equation is consistent with the idea of the potential V being zero at r = infinity.

Superposition states that the above equation can be used once for each of several charges, in order to find the magnitude of the potential V at some location in the vicinity of those charges. No "vectorization" is required; just as with electric potential energy U, one must simply keep track of the sign (+ or –) for each result, and then add them algebraically.

In mathematical terms, using superposition to find the potential V at some precise location due to i charges could be represented by:

$$V = k\sum_{n=1}^{i}\frac{q_i}{r_i} \qquad OR \qquad V = \frac{1}{4\pi\epsilon_o}\sum_{n=1}^{i}\frac{q_i}{r_i}$$

It should be realized that the above characterization is more of an idea than it is an equation.

Everything in the previous discussion has related to the potential V due to one or more POINT charges. (It will be demonstrated in class how those ideas can be expanded to also apply to charged spheres and shells.) Integral calculus, however, must be used to find the potential V due to continuous charge distributions, such as lines of charge or planar sheets of charge. The electric potential V due to a continuous charge distribution is found using:

$$V = k \int \frac{dq}{r} \qquad OR \qquad V = \frac{1}{4\pi\epsilon_o} \int \frac{dq}{r}$$

where, as we've seen before, dq is the charge on a tiny element of the charge distribution and r is the distance from the tiny element to the point in question. Put into words... We will be breaking a continuous charge distribution into many tiny elements of charge dq, then treating each dq as a point charge, then finding the tiny contribution each dq makes to the potential at a specific location (dV), and finally adding up all of the dV results to get our final potential V. (Whew!) This, too, will be demonstrated in class.

3.3 Electric Potential Difference ΔV

It takes work W to move a charge q between points having different potentials V. In particular, work W is always done BY the $\boldsymbol{E}$ field in moving a charge q through a potential difference ΔV. If the work W done by the $\boldsymbol{E}$ field is (+), the charge's potential energy U decreases and, often, its kinetic energy K increases. This is similar to an elevated mass m being dropped in a gravitational field $\boldsymbol{g}$: The field does (+) work W, the mass's potential energy U decreases as it falls, and its kinetic energy K increases. Here is how you know if ANY field is doing (+) work W: If you are holding something at rest in the field and, in order to move it from Point A to Point B, all you have to do is let it go and "VOOP! There it goes!"…then the field is doing (+) work W. We are most interested in $\boldsymbol{E}$ fields here, but the thinking applies to all fields.

On the other hand, when the work W done by the $\boldsymbol{E}$ field is (–), then the charge's potential energy U increases. This is analogous to when a gravitational field $\boldsymbol{g}$ does (–) work as a mass is lifted upward; hopefully, you accept the fact that, in such a case, the potential energy U increases. When an $\boldsymbol{E}$ field does (–) work W on a charge q, the action of an external agent is required to move the charge, because a charge won't spontaneously undergo an increase in electric potential energy U…just like a mass won't spontaneously undergo an increase in gravitational potential energy. An $\boldsymbol{E}$ field does (–) work W to move a charge from Point A to Point B if you have apply a force in order to get the job done.

The relationships mentioned above between work W and potential energy U follow directly from the work W done by a conservative force being equal to the (–) of the change in potential energy U, namely...

$$W = -\Delta U \qquad or \qquad \Delta U = -W$$

...which is a result you hopefully remember from your study of mechanics.

Also, as an aside...Hopefully, too, you recall that only conservative forces (e.g., gravity, the elastic force in a spring, and the electric force) have a potential energy function associated with them, specifically a function of position x, i.e.,

$$F(x) = -\frac{dU}{dx}$$

In any case, here, the conservative force in question is the electrostatic force, which is inextricably connected with the electric field $\boldsymbol{E}$.

The **electric potential difference** (or just **potential difference**, and sometimes simply **voltage**) ΔV between two points has a magnitude equal to the work W per unit charge q that is required to move charge between those two points. The ΔV has the opposite sign of that work W, but the same sign as the change in potential energy ΔU, namely:

$$\Delta V = -\frac{W}{q} = \frac{\Delta U}{q}$$

What the above equation says is this: The bigger the charge q, the more work W is done and the more the potential energy U changes by…for the charge q to move through a given potential difference ΔV.
You might wonder why we even bother with potential difference ΔV: Why not just deal with electric potential energy U? Why bother dividing by the charge q after measuring the work W? It seems like we're taking an extra step that we don't need to take.

Well, here's an analogy that might help you understand; it relates the abstruse concepts of electricity to the more familiar concepts of mechanics...

Potential difference ΔV is analogous to a change in elevation Δh. You should have learned in earlier physics studies that it takes more work to lift a large mass upward for a given change in elevation Δh than it does to lift a small mass upward the same Δh. Furthermore, the change in gravitational potential energy is proportionately larger for the larger mass.

But if you are constructing a multi-story building, it's far more useful for you to know the changes in elevation that need to take place. You might not be concerned with the amounts by which the potential energies of the concrete, wiring, windows, etc. change as you raise them from ground level to the proper stories as the building progresses. In the back of your mind, you'll know that the more massive things will undergo larger changes in potential energy U and the less massive things will undergo smaller changes in U, but you're more interested in making sure that everything makes the correct change in elevation Δh.

In the same way, the potential difference ΔV is a highly useful quantity in physics. It gives us an easy way to deal with charges that move from one location to another, especially if those amounts of charge have varying magnitudes and signs. That is, we can move all manner of charges through a given potential difference ΔV (like moving all manner of masses through a given elevation change Δh) without having to calculate the energy U of each charge, which is UNIQUE to each different charge…just as the potential energy U of every mass is unique to that specific mass.

Besides the equation above that relates ΔV to W and ΔU, the potential difference ΔV between two points A and B can be expressed another way. When there is an electric field ***E*** in the region of space containing points A and B (and there generally is), the potential difference ΔV between A and B can be found using:

$$\Delta V = -\int_A^B \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$$

What the above equation means is this: At each point along a line between A and B, multiply the component of the ***E*** field that is parallel to the line at that point by the tiniest of distances *ds*, all along the line, from A to B. You could think of following the path of an ***E*** field line that goes from A to B, breaking it up into little *ds* pieces as you go. Then, add up all of the ***E*** *ds* products. Then change the sign of that sum…That's ΔV, i.e., the amount by which the electric potential V will have changed from whatever it was at A to whatever it is at B.

The important thing about the equation just discussed is that we don't have to have a charge q, move it from A to B, measure the work W done, and then divide by q…in order to find ΔV. All we need to know is what happens with the ***E*** field between A and B, and we can calculate ΔV. ☺

3.4 Equipotential Surfaces (or Equipotentials)

An **equipotential surface** (or just **equipotential**) is a (usually imaginary, and often curved) surface where all points on the surface have the same electric potential V. While it takes work to move charges between any two locations of differing potential V (as discussed in the previous section), it takes ZERO work to move charges around on a given equipotential. A given equipotential has a fixed potential V, e.g., "This equipotential has a potential V of 2.3 volts, that one has a V of 3.5 volts..." and so on.

An equipotential surface is like a page in a book: It takes no work to move charges around on any given page, even if the page is curved, but it DOES take work to move charges from one sheet to another. Perhaps it won't surprise you that, if a charge q starts on a sheet in the middle of the book, depending on whether it is a (+) or (–) charge AND whether it moves to a different sheet that is nearer the front or nearer the back, the work W required to move q could be either (+) or (–)...but it is definitely NOT zero. A related point is that the potentials V of the "equipotential pages" get more (+) going in one direction and more (–) going in the other. When W is (+), both the change in potential energy ΔU and the change in potential ΔV will be (–)...and vice-versa, as was discussed previously in Section 3.3.

Another analogy for equipotentials is the various layers of an onion. Each onion layer is an equipotential, having a single value of potential V at all points. (You see that, in this case, the equipotential surface is definitely a curved surface.) It takes ZERO work to move charges from any point in a given onion layer to any other point within that layer, but it DOES take work to move charges between layers.

Because the electrostatic force is a conservative force, the work W done when moving a charge q from one equipotential surface to another is independent of the precise locations on each surface where q started and where it stopped. The work W done AND the change in potential energy ΔU AND the potential difference ΔV are also independent of the precise path q takes to get from where it started to where it stopped. All that matters is: On which equipotential did it start, and on which equipotential did it stop?

Using the "equipotential pages" analogy, it takes ONE specific amount of work W to move a charge q from page 83 to page 57 in our "equipotentials book." It doesn't matter WHERE on pages 83 and 57 the charge q started and stopped, and it doesn't matter if the charge went straight from 83 to 57, or if it went from 83 to 24 and then to 57, or if it went from 83 to Timbuktu to Siberia and finally to 57. Start = page 83; stop = page 57. *"How?" "And where on 83?" "Where on 57?"* Nobody cares. Doesn't matter. End of story.

How-closely-spaced the equipotentials of constant incrementation are (e.g., 5 V, 10 V, 15 V, 20 V, etc.) indicates the relative strength of the electric field E through that region of space. Spaced-closely-together equipotentials indicate a strong $\boldsymbol{E}$ field; spaced-far-apart equipotentials indicate a weak $\boldsymbol{E}$ field. Furthermore, electric field lines are always perpendicular to equipotential surfaces, and the $\boldsymbol{E}$ field lines point from the more-positive (i.e., higher) equipotentials toward the more-negative (i.e., lower) potentials. This naturally means that (+) charges are forced from higher toward lower potentials, and that (–) charges are forced from lower to higher potentials.

Here are two examples to emphasize the point that equipotential surfaces are forever perpendicular to $\boldsymbol{E}$ field lines. The first is: For point charges and spherical charge distributions, whose non-uniform electric field $\boldsymbol{E}$ is represented by $\boldsymbol{E}$ field lines that radiate away from the charge (for $+q$) or toward the charge (for $-q$), equipotentials are concentric spheres going away from the charge, with the charge at the center. The second is: Between oppositely-charged parallel plates, where the uniform $\boldsymbol{E}$ field is represented by evenly-spaced, parallel arrows pointing from (+) to (–), equipotential surfaces are equally-spaced planes, parallel to the plates. In both cases, you should be able to see that equipotentials are everywhere perpendicular to $\boldsymbol{E}$ field lines. There are no exceptions to this rule.

Said again: No component of an ***E*** field exists parallel to an equipotential surface; that is, no ***E*** field line – nor any part/component of an ***E*** field vector – "skims the surface" of an equipotential, in any direction.

The last equation in Section 3.3 deserves just a bit more commentary. That equation was:

$$\Delta V = -\int_A^B \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$$

Recall that we said that the potential *V* was constant over the entirety of any particular equipotential surface and that the electric field ***E*** – at every point – is perpendicular to that surface. The above equation supports these ideas. If we are determining the change in potential *ΔV* in going from point A on an equipotential to another point B on the SAME equipotential, every *d**s*** we traverse will be PERPENDICULAR to the ***E*** field at that location; thus, the dot product of ***E*** and *d**s*** will be ZERO at every increment of the path, and then obviously their sum (which is required by the integral sign in the equation) will also be…zero.

And finally, if the electric field ***E*** is constant between two points A and B that are on, you might say "adjacent equipotentials," the above equation reduces to:

$$\Delta V = -\int_A^B \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -E\int_A^B ds = -Es$$

where *s* is the distance of the perpendicular between the equipotentials containing A and B.

This last result is exactly the equation to use if we want to find the magnitude of the ***E*** field between the parallel plates of a charged capacitor. Each plate of a charged capacitor is an equipotential surface, having its own unique potential *V*. For capacitors, the *s* in the above equation turns out to be nothing more than the separation between the capacitor plates. (Sometimes it's called *x* instead, or maybe *d*.) We will investigate this idea more, later on.

3.5 The Mathematical Relationship Between E and V

Assume we can write a mathematical equation *V(x, y, z)*, where the potential *V* throughout a given region of space is known as a function of the coordinates *x*, *y*, and *z*. Then the electric field function ***E*** in any specific direction (*x*, or *y*, or *z*) can be found by taking the negative **partial derivative** of *V(x, y, z)* with respect to that direction. That is:

$$\overrightarrow{\boldsymbol{E_x}} = -\frac{\partial V}{\partial x} \quad and \quad \overrightarrow{\boldsymbol{E_y}} = -\frac{\partial V}{\partial y} \quad and \quad \overrightarrow{\boldsymbol{E_z}} = -\frac{\partial V}{\partial z}$$

You might want to take ten seconds right now to glance back at the last equation included in Section 3.3, as well as the equations in Section 3.4, and here's why: All of those equations have exactly the same meaning as what is printed above.

Anyway, the symbol above that looks like a fancy backwards 6 is the partial derivative symbol. We use these instead of the familiar *d* used in conventional derivatives like you've seen before, as in...

$$g(x) = \frac{d}{dx} f(x) = \frac{df}{dx}$$

Partial derivatives come into play when we have a function having several variables, rather than just a single variable. In earlier physics studies, you learned about the relationships between position ***x***, velocity ***v***, and acceleration ***a***, all of which were often given as functions of a single variable: time *t*. Thus, there was only one variable, time *t*, to differentiate (or integrate) with respect to. We could differentiate or integrate one or more times, but always with respect to only *t*.

However, suppose we have a function of several variables, say a fictitious function G that is a function of the Cartesian coordinates x AND y AND z. If we decide we want to differentiate G, then we have the choice to differentiate it with respect to any of the three variables x OR y OR z.

Taking partial derivatives is no more difficult than taking conventional derivatives. For example, in taking the partial derivative of a function with respect to x, any term in the function that has an x (there might be more than one) must be differentiated in the standard way. Any term without an x is considered a constant, and thus has a derivative of zero. For example, using our fictitious function G, if:

$$G(x, y, z) = 5x^3y^2z - 2xy + 8y^3z^2 + 3x^2z^3 \qquad \text{then}$$

$$\frac{\partial G}{\partial x} = 15x^2y^2z - 2y + 6xz^3$$

And while taking partial derivatives isn't too difficult anyway, once you know what you're doing, obtaining the electric field function ***E*** from the electric potential function *V* is sometimes simpler than you might expect. That is, you might NOT have to take a partial derivative to find ***E***, based on *V*. For example, the potential function *V* might be given in terms of ONLY *x*, i.e., you might be given only *V(x)* rather than *V(x, y, z)*, and then, instead of taking...

$$\overrightarrow{\boldsymbol{E_x}} = -\frac{\partial V}{\partial x} \quad and \quad \overrightarrow{\boldsymbol{E_y}} = -\frac{\partial V}{\partial y} \quad and \quad \overrightarrow{\boldsymbol{E_z}} = -\frac{\partial V}{\partial z}$$

...you'll just have to take...

$$E_x = -\frac{dV}{dx}$$

...where the direction of the ***E*** field, obviously, will be the *x*-direction.

3.6 Electric Potential V and Conductors in Electrostatic Equilibrium

Back in Section 2.7, we discussed some properties of a conductor in electrostatic equilibrium, namely:

1. The electric field ***E*** is zero everywhere inside the conductor.
2. Any net charge q on the conductor resides entirely on its outermost surface.
    a. On a conducting sphere, the net charge spreads uniformly over the surface, producing a uniform surface charge density σ.
    b. On an irregularly-shaped conductor, the surface charge density σ is greatest where the radius of curvature is the smallest, i.e., at sharp points.
3. The electric field ***E*** at the outer surface is everywhere perpendicular to the conductor's surface.

To these three, we will now add that when a conductor in electrostatic equilibrium has an excess charge q, the entire surface and entire volume of the conductor is at a uniform electric potential V. This is true even if the conductor has a cavity within it. One might call the entire charged conductor that is in electrostatic equilibrium an "equipotential volume."

Finally… Suppose we have an isolated conducting object that is in the presence of an external electric field ***E***, i.e., an ***E*** field that is being generated by some external agent. (For now, let's assume our conductor has zero net charge.) The external ***E*** field will momentarily have an effect on the mobile charges at the conductor's surface. Those charges will then re-distribute themselves in a non-uniform fashion, such that the ***E*** field generated BY those mobile surface charges exactly cancels (within the conductor) the external ***E*** field. And it wouldn't matter if our conductor DID have a net charge; its re-distributed surface charges would still cancel (within the conductor) the external ***E*** field. The net ***E*** field below the conductor's surface will remain zero...and the electric potential V everywhere throughout the conductor will have some constant (and often nonzero) value.