<u>Electric Fields and Gauss's Law</u>

2.1 The Electric Field E

Every electric charge sets up an **<u>electric field</u>** ***E*** in the space around it. The electric field ***E*** is a set of vectors (like tiny arrows), one at every location in the space around an electric charge. Each vector points in some particular direction, based on the distribution of the charge(s) that create(s) the field.

If some other charge q is brought into the region where there is already an electric field ***E*** set up by a pre-existing charge distribution, then q will experience a Coulombic force ***F*** due to those already-there charges. Said another way: The force ***F*** acting on q at any given point is due to the electric field ***E*** that already exists at that location, due to the other charges in the vicinity. The magnitude and direction of the force ***F*** on q will depend on where q is located within the ***E*** field, as well as on the magnitude and sign of the charge q. It is important to realize that q does not experience any force due to any ***E*** field that IT might create; only ***E*** fields contributed by OTHER charges exert the force ***F*** on q.

Incidentally, the reason conductors conduct is that, when an ***E*** field is impressed through the material, conduction electrons in the outer energy level of each atom begin moving through the crystal lattice. This is what happens in electric circuits. In insulators, the energy required to get any electrons flowing is prohibitively large; ***E*** fields of ordinary magnitudes don't exert enough force on the electrons to get them to flow, *en masse*, through the material.

2.2 Mathematical Definition of the Electric Field E

The magnitude of the electric field ***E*** at any point in space is equal to the electric force ***F*** that some charge q would experience if it were at that point, divided by the charge q. Because the direction of the force on q

depends on whether q is (+) or (−), a convention was established about the direction of the $\boldsymbol{E}$ field vector. That convention is that the $\boldsymbol{E}$ field vector at each location always points in the direction of the force that a $+q$ would experience, at that location. In vector terms, we thus define the electric field vector $\boldsymbol{E}$ as:

$$\vec{\boldsymbol{E}} = \frac{\vec{\boldsymbol{F}}}{q}$$

It follows from the above discussion that positive charges in an $\boldsymbol{E}$ field are forced in the direction of the $\boldsymbol{E}$ field; negative charges are forced opposite the $\boldsymbol{E}$ field.

One of the most important equations in electrostatics is the electric force $\boldsymbol{F}$ on a charge q that is in the presence of an electric field $\boldsymbol{E}$; it is same equation as above, but written in an easier-to-remember fashion:

$$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}}$$

$\boldsymbol{F} = q\boldsymbol{E}$ exactly corresponds to the description given above, in Section 2.1. Furthermore, the force $\boldsymbol{F}$ is the same electric force $\boldsymbol{F}$ that can be found from Coulomb's law; which of the two equations you use just depends on the information you are given in a certain problem. As with Coulomb's law, don't state a force with a (−) sign; just calculate the magnitude and then figure the direction based on the given information.

One more word about $\boldsymbol{F} = q\boldsymbol{E}$... It is exactly analogous to the gravitational force $\boldsymbol{F_g}$ on a mass m in a gravitational field $\boldsymbol{g}$, namely:

$$\overrightarrow{\boldsymbol{F_g}} = m\vec{\boldsymbol{g}}$$

By the principle of superposition, the $\boldsymbol{E}$ field's magnitude and direction at any location – due to a distribution of several charges – is the vector sum of the $\boldsymbol{E}$ fields generated by the individual charges in the distribution.

2.3 Electric Field Lines

Electric field lines are a convenient way to visualize the direction and magnitude of the electric field ***E*** in the space surrounding particular charge distributions.

Recall that ***E*** field vectors can be visualized as the billions of tiny arrows that fill the entire space around some field-creating distribution of charge. Consider the tip of a single one of those arrows: It points to the tail of exactly one other arrow. And THAT arrow's tip points to the tail of exactly one other arrow. And so on. You could draw a smooth line connecting our first ***E*** field vector to the second, and then to the third, and so on. Then you could begin again with some other vector in the region, drawing another smooth line through an aligned sequence of arrows. Repeat this process a number of times and, pretty soon, you'll have drawn a bunch of electric field lines throughout the space surrounding the charge distribution that is responsible for the creation of the ***E*** field.

Thus, we can say the following four things about electric field lines:

1. ***E*** field lines are tangent to each ***E*** field vector-arrow at every location in the space around a particular charge distribution.

2. ***E*** field lines originate on (+) charges and point away from them. ***E*** field lines point toward (−) charges and, in fact, terminate on them. The larger the magnitude of a particular charge q, the greater the number of ***E*** field lines that originate (or terminate) on it. Thus, a charge of $+2q$ would have twice as many ***E*** field lines that originate on it as would a charge of $+q$; a charge of $-3q$ would have three times as many ***E*** field lines that terminate on it as would a charge of $-q$.

3. Because each ***E*** field vector points to only one of its nearest-neighbor vectors, it logically follows that our continuously-extending-through-space ***E*** field lines never touch each other, nor do they cross each other, nor do they ever get tangled up. Each ***E*** field line is a smooth, undisturbed, and uninterrupted continuum, originating on some (+) charge and terminating on some (−) charge.

4. ***E*** field lines represent lines of force; that is, for a charge q that is IN an ***E*** field, the direction of the electric force ***F*** on that charge q is parallel/tangent to an ***E*** field line. It is important you realize that, in general, ***E*** field lines do NOT represent the trajectory that a charge q within the ***E*** field would follow. In many situations, ***E*** field lines curve; you should have learned in your study of mechanics that, for a particle to follow a curved path, there must be a net force on it that is perpendicular to its velocity vector...not parallel to it. So, once again, ***E*** field lines do NOT represent the trajectory of a charge q within an ***E*** field.

E field lines bunched close together indicate a strong electric field; spread-out, sparsely-spaced electric field lines indicate a weak electric field.

We will explain in the near future how to invent imaginary surfaces (we will call them Gaussian surfaces) that we mentally construct to exist in the space surrounding electric charges. We will also need to be able to imagine the ***E*** field lines "poking through" (or puncturing, or piercing) these Gaussian surfaces. Perhaps you can visualize now that, when the ***E*** field lines are closely spaced, many of them will pierce a "unit area" of a Gaussian surface. At locations where the ***E*** field lines are sparsely spaced, NOT very many of them will pierce an identical "unit area."

2.4 Specific Instances of Calculating the Electric Field Strength

By combining Coulomb's law with the definition of the electric field ***E***, i.e.,

$$F = k\frac{q_1\, q_2}{r^2} = \frac{1}{4\pi\epsilon_o}\frac{q_1\, q_2}{r^2} \qquad and \qquad E = \frac{F}{q} \quad (i.e., \quad F = qE\ )$$

it is easily shown that the magnitude of the ***E*** field a distance *r* away from a point charge *q* is given by:

$$E = k\frac{q}{r^2} = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}$$

The direction of the ***E*** field produced by a point charge *q* definitely DOES depend on the sign of the charge *q*, but you should NOT insert a sign when using the above equation. We said earlier that ***E*** field lines originate on (+) charges and terminate on (–) charges. Thus, if *q* is (+), the ***E*** field points away from *q*; if *q* is (–), the ***E*** field points toward *q*. And at a distance *r* away from the point charge *q*, the ***E*** field's magnitude (i.e., its strength) is given by the above equation.

For a point charge *q*, the ***E*** field lines are perfectly straight, extending in every direction, either away from (for +*q*) or toward (for –*q*) the point charge. The ***E*** field created by a point charge *q* is a prime example of a **non-uniform *E* field** because the strength of the field very much depends on the point in space we are talking about. Close to the charge *q*, the ***E*** field is strong; the ***E*** field lines are bunched closely together. The farther away from the charge we go, the more spread out the ***E*** field lines get, and the weaker the field becomes. You see that this description is supported by the equation at the top of this page: As *r* increases, ***E*** decreases.

In a **uniform *E* field**, on the other hand – which is NOT produced by a point charge! – the ***E*** field's strength is the same everywhere. A uniform ***E*** field is modeled by ***E*** field lines that are parallel to each other, and evenly spaced, throughout the entire region...not densely-spaced in some places and sparsely-spaced in others, as is the case for a point charge q.

A uniform ***E*** field is set up between a pair of large, oppositely-charged, and parallel conducting plates (which is typically what we have with a capacitor; story for another day...). Maybe this will help: Imagine a "ceiling" of (+) charge and a "floor" of (−) charge. The ***E*** field within the "room" is directed straight downward, from the (+) ceiling to the (−) floor, like evenly-spaced streamers hanging straight downward throughout the room. THAT'S kind of like a uniform ***E*** field.

Superposition states that the equation $E = k\frac{q}{r^2} = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}$ can be used once for each of several charges q_1, q_2, ... in order to find the ***E*** field's magnitude and direction at a specific location in the vicinity of those point charges. The superposition process for ***E*** fields isn't necessarily easy to carry out because of the vector nature of ***E*** fields but, with enough time and patience and attention to detail, it can be done.

While there is a simple formula for finding the magnitude of the electric field ***E*** at any given distance r away from a point charge q (i.e., the one in the previous paragraph!) integral calculus must be used to find the ***E*** field due to **<u>continuous charge distributions</u>**. Examples of continuous charge distributions are (1) a line of charge and (2) a **<u>surface charge</u>** distributed over a flat (i.e., planar) surface. The electric field ***E*** at some location due to such a continuous charge distribution can be found using:

$$\vec{\boldsymbol{E}} = k\int\frac{dq}{r^2}\,\hat{\boldsymbol{r}}$$

where dq is the charge on a tiny element of the charge distribution and r is the distance from the tiny element dq to the point in question. Essentially, we are treating the tiny charge element dq as a point charge, finding its tiny contribution to the ***E*** field at that location (dE) and adding up all the little dE

results from each of the dq point charges. (That's why the integration sign is in the equation!) The familiar $\hat{\boldsymbol{r}}$ is a unit vector pointing from dq to the point/location in question. When integrating with this equation, omit the $\hat{\boldsymbol{r}}$; you will simply integrate r to the power of –2.

2.5 Electric Flux Φ_E and Gaussian Surfaces

The concept of **electric flux** Φ_E quantifies the notion of "the number of ***E*** field lines piercing a surface." The electric flux Φ_E passing through any surface (real or imaginary) is defined as:

$$\varphi_E = \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}$$

A **Gaussian surface** is a surface that is pierced by electric field lines. Typically, Gaussian surfaces are imaginary: They are mentally-constructed and "placed" in certain, specific locations. The reason we do this is because the surfaces help us calculate the strength of the ***E*** field at those locations, which are always some distance away from a charge distribution that is generating an electric field ***E***.

Now, here is something weird...

Note in the previous equation that the ***dA*** has a vector-arrow over it. ("Area as a vector? What?!?!") That's right; for the purposes of the above electric flux Φ_E equation, physicists have decided that any area ***A*** (or area element ***dA***) will be a vector quantity. That fact, while weird, isn't that big of a deal. ("Okay, area can be a vector. Whoop-dee-doo...") However, here IS the big deal: The direction of any area vector is taken to be perpendicular (i.e., normal) to the surface of the area element. It's like an archer's arrow embedded into a wooden shield, so that it sticks out normal to the surface of the shield; no matter which way you tilt the shield, the arrow tilts right along with it, always sticking out at perfect right angles to the shield. So, from now on, whenever you see the symbol $d\boldsymbol{A}$ or $\boldsymbol{A}$, you need to imagine a wooden shield (that's the dA or the A) with an arrow sticking out at right angles to it; the arrow represents merely the DIRECTION assigned to the $d\boldsymbol{A}$ or $\boldsymbol{A}$ vector.

As is indicated by the dot-product nature of the equation for electric flux Φ_E, we are interested only in the component of electric flux Φ_E that is parallel to each $d\boldsymbol{A}$ vector (i.e., parallel to each arrow that is sticking out of each $d\boldsymbol{A}$ shield). And because the direction of each little area vector $d\boldsymbol{A}$ (i.e., the arrow) is perpendicular to the surface element itself (i.e., the shield), another way of saying the same thing is: "We are interested only in the component of electric flux Φ_E that is perpendicular to each $d\boldsymbol{A}$ element's surface-shield." An analogy to help you understand flux Φ_E is that all the surface area elements $d\boldsymbol{A}$ taken together (i.e., the Gaussian surface) comprise an archer's target and the $\boldsymbol{E}$ field lines are the arrows; the flux Φ_E is nonzero when arrows are able to become embedded into the target, either straight-on or at some angle.

Correspondingly, any flux Φ_E that has a direction parallel to a real or imaginary Gaussian surface (that is, that skims the surface) is of no interest. It's like you are getting ready to shoot an arrow at a distant target and, right as you take aim, somebody rotates the target 90^o so you can only see it edge-on. You won't hit the target; there will be zero "flux" that passes through the target-surface.

We will use only very basic applications of the electric flux Φ_E equation; namely, we will choose Gaussian surfaces such that the $\boldsymbol{E}$ field magnitude is the same everywhere on the surface. In doing so, the $\boldsymbol{E}$ in the above equation will be a constant and can thus be moved outside the integral. Furthermore, the surfaces will have areas described by simple area formulas, like for a sphere or a cylinder, and so summing all the little dA elements just means finding the total surface area with those easy formulas. In short, as long as the ***E*** field lines are everywhere perpendicular to the tiny dA elements that they pierce, the above equation boils down to:

$$\varphi_E = \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{A}}$$

We must be careful, however. Even if the ***E*** field is constant, suppose that the ***E*** field lines pierce the surface at an angle other than along the perpendicular. In that case, the angle θ between the ***E*** field lines and the normal to the surface (Remember the arrow sticking out of the wooden shield?) comes into play, along with the definition of the dot product:

$$\varphi_E = \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{A}} = EA\cos\theta$$

You might recall the normal line that you met in earlier physics studies in optics, where angles of incidence θ_i and angles of reflection or refraction θ_r were all measured – NOT relative to the optical boundary – but instead relative to the NORMAL to the boundary. We have the same situation here: θ is the angle between the NORMAL to the surface (i.e., the arrow sticking out of the shield) and the incoming (or outgoing) ***E*** field lines.

Here is another analogy to help you fully grasp the meaning of the term flux...

Imagine you are standing shoulder-deep in a river, holding a hula hoop (with an opening of area A) beneath the water's surface. How you orient the hula hoop beneath the surface (i.e, what θ you use) will determine how much water flows through the hoop in a given time, and the amount of water per time is analogous to the flux Φ_E. If you hold the hoop vertically, with the plane of the hoop perpendicular to the water current, you will maximize the amount of "water-flux" that passes through the hoop. If you hold the hoop horizontally under the water, you will get ZERO water-flux through the hoop. And, obviously, intermediate angles-of-hoop-tilt will yield intermediate amounts of water-flux.

Two more notes about electric flux Φ_E:

1. Whether the ***E*** field lines are incoming or outgoing through a surface determines the sign (+ or –) for electric flux Φ_E. We will discuss this more later but, by convention, outgoing ***E*** field lines yield a $+\Phi_E$ and incoming ***E*** field lines yield a $-\Phi_E$.

2. When the Gaussian surface is a 3-D surface (like a hollow sphere or a hollow, capped cylinder), ONLY charge that is enclosed/within/on-the-inside-of the volume surrounded by the surface generates a NET flux Φ_E through the surface. Specifically, if the enclosed charge is (+), a net $+\Phi_E$ passes OUTWARD through the surface. If the enclosed charge is (–), a net $-\Phi_E$ passes INWARD through the surface. Any charges outside the surface generate zero net Φ_E over the entire surface because the ***E*** field lines originating on (or terminating on) those "outside" charges will pierce the surface at some location on the way IN (that's $-\Phi_E$), but will also pierce the surface again (at some other location) on the way OUT (that's $+\Phi_E$). While at "this point" or "that point," an outside charge WILL produce some amount of localized, nonzero electric flux Φ_E, the NET flux Φ_E through the entirety of the 3-D surface due to those outside charges will be...zero.

2.6 Gauss's Law

Gauss's law is a useful tool that allows us to determine the magnitudes of electric fields *E* around particular distributions of charge. These distributions might be point charges, but usually they are more-complicated distributions. (The magnitude of the ***E*** field at any distance *r* from a point charge *q* is easily found using the equation given in Section 2.4, so why bother using Gauss's law for that?) Gauss's law is especially useful when the charge distribution (and thus the electric field ***E*** generated) exhibits a high degree of symmetry: spherical, cylindrical, or planar symmetry.

Spherical symmetry will apply whenever we have single point charges, charged spheres or shells, or concentrically-placed charged spheres or shells. Cylindrical symmetry will apply whenever we have a long line of charge, charged cylinders or tubes, and concentrically-placed charged cylinders or tubes. Planar symmetry will apply whenever we have large (ideally, infinitely large) charged sheets.

Gauss's law says that the net electric flux Φ_E through any closed Gaussian surface equals the net charge q_{in} (i.e., its algebraic sum) inside the surface, divided by the constant ε_o:

$$\varphi_E = \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{q_{in}}{\epsilon_o}$$

where, as before,

$$\epsilon_o = 8.85 \, x \, 10^{-12} \, \frac{C^2}{N \cdot m^2}$$

In essence, Gauss's law states that the number of ***E*** field lines piercing any 3-D closed surface depends only on the net charge q_{in} enclosed by (i.e., inside) that surface. Sometimes, to emphasize the need to consider all charge within a Gaussian surface, Gauss's law is written:

$$\varphi_E = \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{\sum q_{in}}{\epsilon_o}$$

You will need to understand how to use Gauss's law for various types of charge distributions, so we will go through several examples in class.

2.7 Conductors in Electrostatic Equilibrium

Recall that electrostatic equilibrium simply means that charges are no longer moving; it has NOTHING WHATSOEVER to do with whether or not the net charge on an object is – or isn't – zero. Objects in electrostatic equilibrium can still have a net charge, which means – you should have learned in chemistry – that they have either gained or lost electrons. When studying electrostatics in this class, you can fairly well count on the objects being charged; there would be very little that was interesting and almost nothing to calculate, otherwise. ☺

A conductor in electrostatic equilibrium has the following properties:

1. The electric field ***E*** is zero everywhere inside the conductor.

2. Any net charge that the conductor has (either + or –) resides entirely on its outermost surface. This is true even if there is a cavity within the conductor, as in the case of a hollow, conducting shell. (There are a few interesting cases that we will discuss in class, but – good news! – these are all very easy to understand, once someone has explained what's going on.) Now, on the exterior of a charged conductor, HOW the net charge distributes itself depends on the geometry of the conductor...

    a. On a conducting sphere (either a solid mass or a hollow shell), the net charge spreads uniformly over the (symmetric) outer surface, producing a uniform surface charge density σ.

    b. On an irregularly-shaped conductor (e.g., a metal statue), the surface charge density σ is greatest where the radius of curvature is the smallest, i.e., at sharp points. The surface charge density σ is least where the radius of curvature is large, i.e., at gradually-sloped or nearly-flat regions.

3. The ***E*** field at the outer surface of a charged conductor is everywhere perpendicular to the conductor's surface, and has no component parallel to the surface. If any component of the ***E*** field WERE parallel to the surface, then that ***E*** field component would obviously apply forces to the mobile charges at the surface, and parallel to the surface. These parallel-to-the-surface forces would then result in those

charges being forced to move along the surface and, by definition, a conductor in electrostatic equilibrium has zero moving charges! So the ***E*** field at the surface MUST be entirely perpendicular to the surface.

The magnitude of the ***E*** field at each point on a conductor's surface is given by the following equation:

$$E = \frac{\sigma}{\epsilon_o}$$

It is extremely important to note that none of the three stated points above apply to an insulator that has been given a net charge. If an insulator is given a net charge, the charge does not spread around on the surface or travel through the main body of the material. The locations of the excess charge are limited to the insulator's surface, and specifically to the points where the insulator was contacted (touched) with the charging material. Thus, there are many charge distributions that are possible for non-conductors because when you put charge in a certain place on its surface, it stays there and doesn't move. Not so with conductors; the excess charge disperses on the surface, depending on the geometry of the conductor.

2.8 Electric Dipoles

An **electric dipole** consists of two point charges of equal and opposite magnitude q that are separated by a distance d. The electric dipole moment ***p*** is a vector having a magnitude of qd, and which points from the $-q$ to the $+q$.

Because the charges in an electric dipole $-q$ and $+q$ are equal and opposite, there is no net force ***F*** on a dipole when it is in a uniform electric field ***E***. Recall that a uniform ***E*** field is one that is constant at all points, both in magnitude and direction, and can be visualized with parallel, straight, and evenly-spaced arrows. However, in a non-uniform electric field ***E***, the forces on the $-q$ and $+q$ charges typically won't cancel because $-q$ and $+q$ (at the ends of the dipole) will be in locations that DIFFER in ***E*** field strength. In such a case, $-q$ and $+q$ will experience forces of DIFFERING magnitudes. The result is that there WILL be a net force $\boldsymbol{F}$ on the two-charge dipole. However, in ALL cases – uniform $\boldsymbol{E}$ field or not – the oppositely-directed forces on $-q$ and $+q$ will tend to produce a torque τ on the dipole, trying to twist it until it achieves stable equilibrium, at which point the dipole no longer has any tendency to twist. A little thought will lead you to the conclusion that stable equilibrium for an electric dipole is reached when the electric dipole moment vector $\boldsymbol{p}$ points (which, you recall, is always directed away from $-q$ and toward $+q$) in the same exact direction as the $\boldsymbol{E}$ field.